DTIC File Copy

(1)

AD-A212 890

CMS Technical Summary Report #89-24

PARABOLIC EQUATIONS FOR CURVES ON SURFACES. II. INTERSECTIONS, BLOW UP AND GENERALIZED SOLUTIONS

Sigurd Angenent

UNIVERSITY OF WISCONSIN

CENTER FOR THE MATHEMATICAL SCIENCES

Center for the Mathematical Sciences
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

January 1989

(Received January 26, 1989)

Approved for public release
Distribution unlimited

Sponsored by

Air Force Office of Scientific Research
Building 410
Bolling Air Force Base
Washington, DC 20332

National Science Foundation
Washington, DC 20550

UNIVERSITY OF WISCONSIN - MADISON
CENTER FOR THE MATHEMATICAL SCIENCES

PARABOLIC EQUATIONS FOR CURVES ON SURFACES. II INTERSECTIONS, BLOW UP AND GENERALIZED SOLUTIONS.

Sigurd Angenent

Technical summary report # 89-24
January 1989

ABSTRACT

In this paper we describe a theory for parabolic equations for immersed curves on surfaces, which generalizes the curve shortening or flow by mean curvature problem, as well as several models in the theory of phase transitions in two dimensions.

We describe a class of equations for which the initial value problem is well posed for rough initial data, for which one can give a description of the way a smooth solution becomes singular, and for which one can define generalized solutions, i.e. solutions which are smooth, except at a discrete set of times.

The methods which are used in this paper are more geometrical than those of part I. By comparing arbitrary solutions with certain special solutions, and by considering the way they intersect, estimates for the curvature and the tangent are derived, which allow one to study the initial value problem, and the way solutions become singular.

AMS (MOS) Subject classifications: 35K15, 53C99
Key words: parabolic differential equation, curve shortening, geometric heat equation.

DEPARTMENT OF MATHEMATICS UNIVERSITY OF WISCONSIN - MADISON MADISON, WI53706

While I was working on this paper, I was partially supported by a National Science Foundation Grant No. DMS - 8801486, an Air Force Office of Scientific Research Grant No. AFOSR - 87 - 0202 and the Netherlands Organization for Pure Scientific Research (ZWO).

CONTENTS

Parabolic Equations for Curves on Surfaces.

II Intersections, blow up and generalized solutions.

Sigurd B. Angenent[1]

Introduction.

In this paper we continue our study of families of closed oriented immersed curves $\gamma(t)$ $(0<t<t_0)$, on an orientable surface (M, g), whose normal velocity $v^\perp$ satisfies

(0.1) $\quad v^\perp = V(t_{\gamma(t)}, k_{\gamma(t)}),$

for some function $V: S^1(M)\times\mathbf{R}\rightarrow\mathbf{R}$. Here $t_{\gamma(t)}\in S^1(M)$ is the unit tangent to the curve $\gamma(t)$, and $k_{\gamma(t)}$ is its curvature.

As we observed in the introduction to part I, this problem is a generalization of the "curve shortening problem," where $V(t,k)=k$, and which has recently enjoyed the attention of a number of people, both for purely aesthetic reasons, and also from a more applied point of view [AL, EW, G, GH, GuA, Gr1, Gr2, RSK].

For the curve shortening problem it is known that solutions become singular in finite time. One expects that this can only happen if the curve shrinks to a point (as in [AL, GH, Gr1]), or if the curve has a self intersection, and one of its "loops contracts."

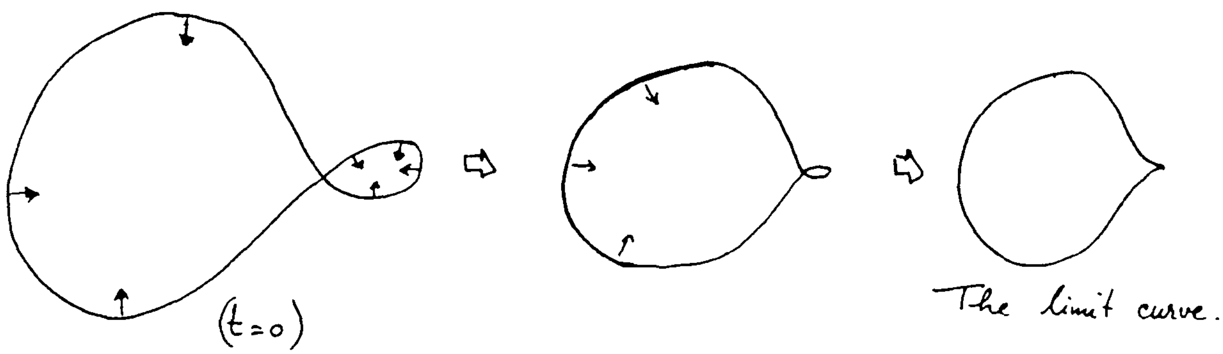

Figure 0.1 — A contracting loop and the limit curve.

By drawing pictures one can convince one self that a limit curve with cusp like singularities must exist, and that there must be a smooth solution of (0.1) with $V\equiv k$, which has this limit curve as initial value. Thus one can imagine that there might be a generalized solution of (0.1), which becomes singular at a discrete set of times, and either exists for ever, or shrinks to a point in finite time. Indeed for the curve shortening problem Grayson [Gr1, Gr2] has shown that an embedded curve never becomes singular, unless it shrinks to a point on the surface.

1. While I was working on this paper, I was partially supported by a National Science Foundation Grant No. DMS-8801486, an Air Force Office of Scientific Research Grant No. AFOSR-87-0202, and The Netherlands Organization for Pure Scientific Research (ZWO).

Our goal in this paper is to prove a rigorous version of these statements, for as large a class of V's as we can.

In part I we found natural conditions on V for which the initial value problem associated with (0.1) has a short time solution for initial data γ_0 with L_p curvature. The methods which we used were mainly based on integral estimates and existing results on parabolic PDE ([Ei, LUS, DPG, A1]). By contrast, the tools of this paper are more picturesque. Most arguments involve comparing a given solution with several special solutions, paying particular attention to their intersections. It turns out that the number of such intersections cannot increase with time, and this fact allows us to get estimates on the tangent and the curvature of a solution. The precise results concerning zeroes of solutions of parabolic equations are stated in the first section.

In sections two and three we complete the proof of what was announced and partially proven as "theorem B" in part I. Thus we prove:

Theorem A. *Let V satisfy*

(V_1) $\quad V:S^1(M)\times\mathbf{R}\to\mathbf{R}$ *is locally Lipschitz,*

(V_2) $\quad \lambda^{-1}\le \dfrac{\partial V}{\partial k}\le\lambda$

(V_3) $\quad |V(t,0)|\le\mu$ *for all* $t\in S^1(M)$,

(V_5^*) $\quad |\nabla^h V|+|k\nabla^v V|\le\nu(1+|k|^2)$,

for some positive constants λ, μ, ν.
Then for any locally Lipschitz γ_0 there's a family $\gamma(t)$ $(0\le t<t_{Max})$ of curves which satisfy (0.1), and which have initial position $\gamma(0)=\gamma_0$.

See theorem 3.2.

Starting in section four we shall also assume that V satisfies the symmetry condition

(S) $\quad V(-t,-k)=-V(t,k)$

for all $t\in S^1(M), k\in\mathbf{R}$. If V satisfies this condition and $\gamma(t)$ is any family of curves satisfying (0.1), then the family $\hat{\gamma}(t)$ which is obtained from $\gamma(t)$ by reversing its orientation also satisfies (0.1). The condition (S) is not only sufficient, it is also a necessary condition for (0.1) to be invariant under orientation reversal. In section four we show:

Theorem B. *Let V be a $C^{2,1}$ function on $S^1(M)\times\mathbf{R}$ which satisfies V_2, V_3, V_5^* and S. Then the initial value problem (0.1) has a short time solution for any initial curve which is C^1 locally graphlike.*

A local homeomorphism $\gamma:S^1\to M$ is said to parametrize a C^1 locally graph like curve, if there's a local homeomorphism $\sigma:S^1\times[-1,1]\to M$ such that

(1) $\quad \gamma(x)=\sigma(x,u(x))$ for some continuous function $u\in C^0(S^1)$ with $|u(x)|\le 1/2$.

(2) $\quad$ The partial derivative $\dfrac{\partial\sigma(x,y)}{\partial y}$ is a continuous function which vanishes nowhere, so that $\sigma|\{x\}\times[-1,1]$ is a C^1 regular curve segment for any $x\in S^1$.

(3) $\quad \dfrac{\partial\sigma(x,y)}{\partial x}$ exists and is a continuous function except at $y=u(x)$.

This definition is slightly different from the one which was announced in part I, when theorem B was stated. The difference lies in the more detailed continuity requirements for the map σ.

Having dealt with the initial value problem, we study the way in which a maximal classical solution of (0.1) can become singular. In part I it was shown that a limit curve always exists, and that it is a locally Lipschitz curve with finite total absolute curvature. In particular, away from a finite number of points it is locally the graph of a Lipschitz continuous function.

Using our extra assumptions V_3^* and S, we show in section five that the limit curve is piecewise smooth:

Theorem C. *Let V be a $C^{2,1}$ function on $S^1(M)\times\mathbf{R}$ which satisfies V_2, V_3, V_3^* and S, and let $\gamma:[0,t_{Max})\to\Omega(M)$ be a maximal classical solution of (0.1). Then the limit curve γ^* of the $\gamma(t)$ is a piecewise C^1 curve, which is $C^{2,\alpha}$ away from its singular points.*

One could try to use this limit curve as initial data for (0.1), with the goal of constructing a global "weak solution." It turns out that the limit curve might contain some "bad parts," such as pieces which are parametrized twice, so that the limit γ^* won't be a C^1 locally graphlike curve. However, one can remove certain "redundant" parts of γ^*, and define a *reduced limit curve* γ_{red}. This is done in section six, where we also show that the reduced limit curve is indeed C^1 locally graph like. Then, in the seventh section we prove

Theorem D. *If $\gamma:[0,t_{Max})\to\Omega(M)$ is a maximal classical solution which becomes singular in finite time, then the limit curve either has less self intersections than any of the $\gamma(t)$'s, or else its total absolute curvature K_* satisfies*

$$K_* \le \lim_{t\to t_{Max}} K(t)-\pi,$$

where $K(t)$ is the total absolute curvature of $\gamma(t)$.

This allows one to show that there exists a global "weak solution" of some kind, which either exists for ever, or in finite time collapses to a point on the surface. In particular, if the initial curve is homotopically nontrivial, then this generalized solution must exist forever. Moreover, the set of times at which the generalized solution is singular (is not a classical solution) is discrete.

Finally, we show that if the initial data γ_0 is a simple curve, without nodes (i.e. $V(t_{\gamma_0},k_{\gamma_0})$ never vanishes on γ_0), then the maximal classical solution either exists for ever, or shrinks to a point, or, in other words, that the generalized solution doesn't contain any singularities.

Notation. We use the same notation as in part I. In particular, $\Omega(M)$ is the space of regular C^1 curves in M, i.e. the space of equivalence classes of immersions $\gamma:S^1\to M$, where γ_1 and γ_2 are equivalent if there is a C^1 orientation preserving diffeomorphism h of the circle such that $\gamma_1=\gamma_2\circ h$. We assume that the surface (M,g) is a complete C^∞ smooth surface with bounded scalar curvature; $S^1(M)$ denotes its unit tangent bundle.

A *classical solution* of (0.1) is a family of curves $\{\gamma(t),\ 0\le t<t_0\}$ which at each $t>0$ has continuous curvature, whose normal velocity satisfies (0.1), and which defines a continuous map $[0,t_0)\to\Omega(M)$.

Section I.3 refers to section 3 of part I, and formula (I.4.3) is of course formula three in section four of part I.

1. Intersections, tangencies and nodes.

Let $\gamma_0\in\Omega(M)$ and $\gamma_1\in\Omega(M)$ be two regular curves in M, parametrised by arclength, and let $P=\gamma_0(s_0)=\gamma_1(s_1)$ be a point in their intersection. The intersection is called *transversal* if the unit tangent vectors $\gamma_0'(s_0)$ and $\gamma_1'(s_0)$ are independent. If they are not, then P is a tangency of the two curves, and one has $\gamma_0'(s_0)=\pm\gamma_1'(s_0)$, depending on the orientations of the two curves. If the two tangents point in the same direction, then we shall call P an *oriented tangency*, otherwise we shall say that P is a *reverse tangency*.

If there are $s_0 \neq s_1$ such that $\gamma_0(s_0)=\gamma_0(s_1)$, then $P=\gamma_0(s_0)$ is a self intersection of the curve γ_0. The phrases transversality, oriented and reverse tangency are defined in the same way as for intersections of different curves.

Throughout this section we shall assume that V is at least $C^{2,1}$.

This extra smoothness will allow us to apply the results from [A2].

Theorem 1.1 *Let $\gamma_0, \gamma_1 : [0, t_0) \to \Omega(M)$ be two solutions of (0.1). Then the set of moments in time t at which $\gamma_0(t)$ and $\gamma_1(t)$ have an oriented tangency is discrete in $(0, t_0)$. Moreover, for any $t \in (0, t_0)$ the two curves $\gamma_0(t)$ and $\gamma_1(t)$ have at most a finite number of oriented tangencies.*

The proof of this theorem, and also the proofs of the other results in this section rely on the following result from [A2].

Proposition 1.2 *Let $u : [x_0, x_1] \times (0, t_0) \to \mathbf{R}$ be a continuous classical solution of*

(0.1) $\quad u_t = a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u$

with $u(x_0, t) \neq 0$ and $u(x_1, t) \neq 0$ for all $t \in (0, t_0)$.
Assume that the coefficients a, b and c satisfy

(i) $\delta \le a(x,t) \le \delta^{-1}$ *for some* $\delta > 0$,

(ii) $a, a_t, a_x, a_{xx}, b, b_t, b_x$ *and c are bounded measurable functions on* $[x_0, x_1] \times (0, t_0)$.

Then, for any $t > 0$ the number of zeroes of $u(\cdot, t)$

$$z(t) = \#\{x \in (x_0, x_1) \mid u(x,t) = 0\},$$

is finite. In addition, if, for some $t_1 \in (0, t_0)$, $u(\cdot, t_1)$ has a multiple zero (i.e. if u and u_x vanish simultaneously), then $z(t)$ drops as t increase beyond t_1.

This result is a refinement of earlier results of Sturm, Nickel, Henry, Matano and Fiedler&Angenent (see [A2] for further references). It has been rediscovered several times, and the oldest reference is probably Sturm's [St][2] which appeared in 1836 in the first volume of Liouville's Journal. In this paper Sturm gives a proof of the proposition, implicitely assuming that all occuring functions are real analytic and that the equation is of the form $u_t = (k(x)u_x)_x + q(x)u$.

One should note that, if $u(\cdot, t_1)$ has no multiple zeroes, for some $t_1 > 0$, then all its zeroes are simple and the implicit function theorem says that the same is true for all t which are sufficiently close to t_1. So, $z(t)$ drops if, and only if, $u(\cdot, t)$ has a multiple zero. Therefore the set of times at which $u(\cdot, t)$ has a multiple zero is discrete in $(0, t_0)$.

Proof of theorem 1.1. Let $t_1 \in (0, t_0)$ be given. Just as in section three, we can find a C^∞ smooth immersion $\sigma : S^1 \times (-1, 1) \to M$ which allows us to represent the curves $\gamma_0(t)$ by a function $y = u(t, x)$, at least for t close to t_1(i.e. $x \to \sigma(x, u(t, x))$ parametrises $\gamma_0(t)$). The graph is oriented in the direction of increasing x, and this orientation should be compatible with the orientaton of $\gamma_0(t)$.

Suppose that at $t = t_1$, $\gamma_0(t)$ and $\gamma_1(t)$ have an oriented tangency. Define

$$\Sigma = \{x \in S^1 \mid \gamma_0(t) \text{ and } \gamma_1(t) \text{ have an oriented tangency at } \sigma(x, u_0(t_1, x))\}.$$

2. I had the good fortune of being told about this reference by both Bernold Fiedler and Bernhard Kawohl. It is remarkable that the time independent versions of this proposition have become well known as Sturmian oscillation theorems, while the parabolic counterparts seem to have been forgotten by most mathematicians.

For each $x \in \Sigma$ there is a maximal subarc of $\gamma_1(t_1)$ which contains the oriented tangency at $\sigma(x, u_0(t_1,x))$, and can be represented as the image under σ of the graph of a C^1 function $v: J \subset S^1 \to (-1, 1)$, where J should be open and connected in S^1 (so J is either an open interval, or all of S^1).

The (uniform) continuity of the tangent to $\gamma_1(t_1)$ implies that there are at most a finite number of such *maximal* subarcs. So there are finitely many C^1 functions $v_k: J_k \to (-1, 1)$ whose images under σ contain all oriented tangencies. After shrinking the domains J_k slightly, if necessary, we can ensure that on ∂J_k one has

$$v_k(x) \neq u(t_1,x) \text{ and } |v_k| < 1$$

and, in addition, v_k is C^1 on $\overline{J_k}$ (i.e. $v_{k,x}$ is bounded on J_k).

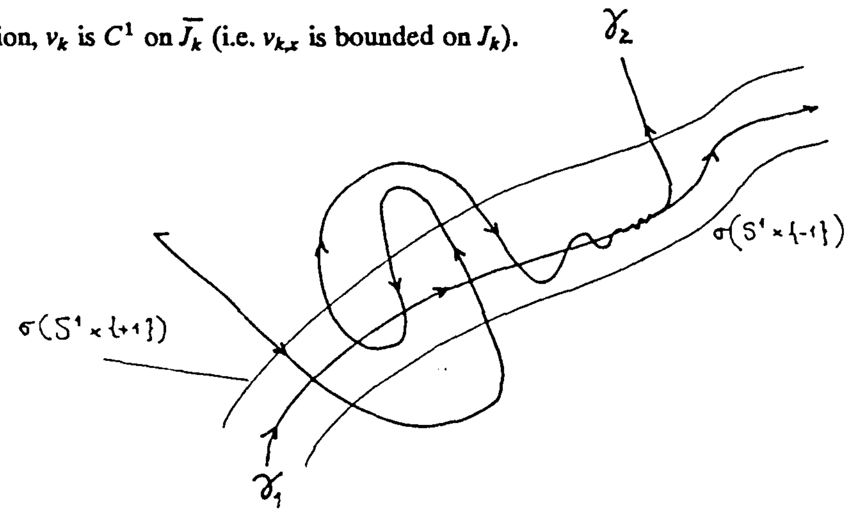

Figure 1.1— Intersection of γ_1 and γ_2.

By continuity of $\gamma_1: [0, t_0) \to \Omega(M)$ the functions v_k can be extended to $(t_1-\delta, t_1+\delta) \times \overline{J_k}$, for some small positive δ, in such a way that this extension parametrizes a subarc of $\gamma_1(t)$ $(t_1-\delta < t < t_1+\delta)$. For t close enough to t_1 the images of the graphs of the v_k contain all oriented tangencies (if there are any) of $\gamma_1(t)$ with $\gamma_0(t)$.

In section I.3 we argued that the equation of motion, (0.1), is equivalent to a parabolic equation of the type

(I.3.2) $\quad u_t = F(x, u, u_x, u_{xx}),$

with F given by (I.3.3). Our assumption that V is $C^{2,1}$ implies that F is also $C^{2,1}$. This implies for any classical solution u of (I.3.2) that the derivatives $\partial_x^{j+2} \partial_t^k u$ with $j+k \leq 2$ exist and are Hölder continuous.

Both u and the v_k are oriented in the direction of increasing x. Therefore they satisfy the same equation (I.3.2), and their difference $w_k = u - v_k$ satisfies a linear equation of the form (0.1), where one puts

$$a(t,x) = \int_0^1 \frac{\partial F}{\partial q}(x, u^\theta, u_x^\theta, u_{xx}^\theta)\, d\theta$$

and $u^\theta = \theta u + (1-\theta) v_k$. For the coefficients b and c one has similar expressions involving other derivatives of F.

The smoothness of the solutions u and v_k which we have just observed suffices to show that the coefficients a, b and c in (0.1) indeed satisfy the hypotheses of the proposition. We have also ensured that the function $w_k(t,\cdot)$ does not vanish on the boundary of J_k, for t close to t_1. Since multiple zeroes of $w_k(t,\cdot)$ correspond to oriented tangencies of $\gamma_0(t)$ and $\gamma_1(t)$, the theorem follows immediately from proposition 1.2. □

Corollary (Unique continuation). *If, at $t=0$, the curves $\gamma_0(t)$ and $\gamma_1(t)$ are different, then they remain different for all $t>0$ for which they are defined.*

Indeed, if they would suddenly coincide for some positive t, then they would have infinitely many oriented tangencies.

A slight variation of the proof of theorem 1.1 will show that a solution $\gamma:[0,t_{Max})\rightarrow\Omega(M)$ of (0.1) has at most a finite number of oriented self tangencies for each $t>0$. The set of t for which such selftangencies occur is discrete in $(0,t_{Max})$.

One can also prove the following by the same kind of arguments.

Variation on theorem 1.1. *Let $\gamma_{1,2}:[0,t_{Max})\times[0,1]\rightarrow M$ be two parametrizations of moving arcs which satisfy (0.1), and for which*

$$\partial\gamma_1(t)\cap\gamma_2(t) = \partial\gamma_2(t)\cap\gamma_1(t) = \phi$$

holds for any $t\in[0,t_{Max})$. Then the set of moments t at which $\gamma_1(t)$ and $\gamma_2(t)$ have an oriented tangency is discrete in $(0,t_{Max})$, while the number of oriented tangencies is finite at any time $t\in(0,t_{Max})$.

In general, theorem 1.1 and its corollaries will not be true for reverse tangencies instead of oriented ones. The point where the proof breaks down is the observation that the functions u and v_k satisfy the same equation (I.3.2) because they are oriented in the same direction. Near a reverse tangency u would be oriented in the direction of increasing x, and v_k would have the opposite orientation. Therefore v_k satisfies (I.3.2) with a different F (namely $F=l^{-1/2}D^{1/2}V(-d\sigma\cdot T,-k)$; cf. (I.3.3)).

However, if the velocity function V satisfies the symmetry condition (S) then these equations coincide. In other words, assuming condition (S), the evolution of a curve does not depend on its orientation. Yet another way to say this is that the evolution according to (0.1) on $\Omega(M)$ is equivariant with respect to the map $\rho:\Omega(M)\rightarrow\Omega(M)$, which sends a curve to the same curve with the opposite orientation.

Since the two equations (I.3.2), corresponding to the two possible orientations one can assign to the graph of $y=u(x)$, coincide when (S) holds, the proof of theorem 1.1 immediately yields the next result.

Theorem 1.3 *Let V be $C^{2,\alpha}$ and assume V satisfies V_2, V_3 and the symmetry condition (S). Consider two solutions $\gamma_0,\gamma_1:[0,t_0)\rightarrow\Omega(M)$ of (0.1), and choose parametrizations of these families of curves (we use the same symbols $\gamma_0,\gamma_1:[0,t_0)\times S^1\rightarrow M$ to denote these parametrizations.)*

Then, at each moment in time t, the number of intersections of $\gamma_0(t)$ and $\gamma_1(t)$,

$$i(t) = \#\{(s_0,s_1)\in S^1\times S^1 \mid \gamma_0(t,s_0)=\gamma_1(t,s_1)\}$$

is finite. This number does not increase with time, and decreases exactly at those t for which $\gamma_0(t)$ and $\gamma_1(t)$ have a tangency (i.e., a nontransversal intersection). The set of such moments t is a discrete subset of $(0,t_0)$.

Just as with theorem 1.1, there is an analogous statement for self intersections of a classical solution $\gamma(t)$ of (0.1): *at any time $t>0$ the curve $\gamma(t)$ has a finite number of self intersections, and this number does not increase with time. It decreass whenever $\gamma(t)$ has a self tangency.*

One can also easily prove a statement analogous to the variation on theorem 1.1.

Variation on theorem 1.3. *Let* $\gamma_{1,2}:[0,t_{Max})\times[0,1]\to M$ *be two parametrizations of moving arcs which satisfy (0.1), and for which*

$$\partial\gamma_1(t)\cap\gamma_2(t) = \partial\gamma_2(t)\cap\gamma_1(t) = \phi$$

holds for any $t\in[0,t_{Max})$. *Then the number of intersections of* $\gamma_1(t)$ *and* $\gamma_2(t)$ *is a finite and nonincreasing function of* $t\in(0,t_{Max})$. *It decreases whenever* $\gamma_1(t)$ *and* $\gamma_2(t)$ *have a tangency.*

Instead of comparing two arbitrary curves, we can take one family of curves $\gamma:[0,t_0)\to\Omega(M)$ and consider the intersections of two infinitesimally close curves $\gamma(t)$ and $\gamma(t+dt)$ in this family. Or, more precisely, we can consider points on $\gamma(t)$ where the normal velocity of the family vanishes. We shall call such a point a *node*.

Theorem 1.4 *Assume that* V *satisfies the conditions* V_1 $(m\geq 2)$, V_2 *and* V_3, *and let* $\gamma:[0,t_0)\to M$ *be a solution of (0.1). Then, for any* $t\in(0,t_0)$, $\gamma(t)$ *has at most a finite number of nodes, and this number does not increase with time. In fact, it drops whenever the normal velocity* $v^\perp$ *of* $\gamma(t)$ *has a multiple zero.*

Proof. Again, we choose an immersion $\sigma:S^1\times(-1,1)\to M$ with which we can represent $\gamma(t)$ as the image of a graph $y=u(t,x)$, at least for all t sufficiently close to some prescribed $t_1\in(0,t_0)$.

The function $u(t,x)$ satisfies (I.3.2), and the nodes of $\gamma(t)$ are in one to one correspondence with the zeroes of $x\to u_t(t,x)$. But $w=u_t$ satisfies a linear equation, as one sees by differentiating (I.3.2) with respect to time:

$$w_t = F_q w_{xx} + F_p w_x + F_u w.$$

As before, we have assumed enough smoothness of F to be able to apply the proposition to w. The theorem then follows immediatly. □

We conclude this section with two special cases, one in which one can state the previous results in slightly different terms, and one in which the proposition 1.2 can be applied in yet another way.

Let V satisfy

$$V(t,0)\equiv 0$$

for all $t\in S^1(M)$, so that the stationary curves for (0.1) are exactly the geodesics of (M,g). Then the nodes of a curve $\gamma(t)$ in a family $\gamma:[0,t_0)\to\Omega(M)$ are exactly the inflection points of $\gamma(t)$, i.e the points where the curvature vanishes. In this case the statement of theorem 1.4 holds for inflection points; at any given moment $\gamma(t)$ has at most a finite number of them, and this number does not increase. An example of a V which satisfies this condition comes from the curve shrinking problem, where $V\equiv k$.

In the other special case we assume that M is a manifold of constant curvature. Recall that a *vertex* of a curve is a point on the curve where the curvature either has a maximum or a minimum (cf [Spivak], in particular page 30).

Theorem 1.5 *If* V *is a function of the curvature only,* $V\equiv V(k)$, *and* M *has constant curvature, then for any solution* $\gamma:[0,t_0)\to\Omega(M)$ *of (0.1), the number of vertices on* $\gamma(t)$ *is finite and nonincreasing with time.*

Proof. We choose an arclength parametrisation $\gamma:[0,t_0)\times\mathbf{R}\to M$ of the family of curves $\gamma(t)$ (so $g(\gamma_s,\gamma_s)\equiv 1$), for which the point with s coordinate $s=0$ moves orthogonally to the curve, i.e. for which $\gamma_t(t,0)\perp\gamma_s(t,0)$. If $L(t)$ is the length of $\gamma(t)$, then $\gamma(t,\cdot)$ is a periodic function, with period $L(t)$.

As a function of s and t, the curvature k satisfies the following equation:

$$\frac{\partial k}{\partial t} = \frac{\partial^2 V(k)}{\partial s^2} + \beta(t,s)\frac{\partial k}{\partial s} + (R + k^2)V(k) \tag{1.2}$$

(cf. [AL] where this equation is stated in the case $V(k) \equiv k$; in our case it can be derived from (I.4.2)). The quantity $\beta(t,s)$ is defined by

$$\beta(t,s) = \int_0^s k(t,s')V(k(t,s'))\,ds'.$$

Differentiation of (1.2) with respect to s leads to a linear equation of the form (0.1) for $u = k_s$. Hence we can apply proposition 1.2 again, or rather the version with periodic boundary conditions, which was also proved in [A2], to complete the proof. □

2. *Local Lipschitz estimates.*

In section nine of part I we defined, for any C^2 curve $\gamma \in \Omega(M)$, the quantity

$$\alpha_\epsilon(\gamma) = sup_{|s_1 - s_0| < \epsilon} \left| \int_{s_0}^{s_1} k(s)\,ds \right|,$$

and interpreted it as the maximal angle between two tangents to the curve, whose base points are closer than ϵ, when measured along the curve. Clearly, since the curve is C^1, one has

$$\lim_{\epsilon \to 0} \alpha_\epsilon(\gamma) = 0. \tag{2.1}$$

We shall now show how the definition of the quantity $\alpha_\epsilon(\gamma)$ can be extended so as to include nonsmooth curves. Consider an absolutely continuous curve $\gamma : S^1 \to M$, which is parametrised by arclength; thus in local coordinates (x^1, x^2) the functions $x^i \circ \gamma$ are absolutely continuous, and the tangent to γ, which exists almost everywhere, has unit length (a.e.) . Just as in the C^1 case, any absolutely continuous curve, with $\dot\gamma \neq 0$ (a.e.) has a continuous reparametrisation in which $|\dot\gamma| = 1$ (a.e.) . Parallel transport along an absolutely continuous curve still makes sense: to transport the vector $\xi^1 \partial_{x^1} + \xi^2 \partial_{x^2} \in T_{\gamma(t_0)}M$, one solves the linear ODE

$$\frac{d\xi^i}{ds} = -\{{}_{jk}^{\,i}\}\, \xi^j\, t_\gamma^k \tag{2.2}$$

where $\{{}_{jk}^{\,i}\}$ are the Christoffel symbols of the metric in the chosen coordinates, and t_γ^k are the components of the unit tangent to γ. Since $\dot\gamma$ has unit length (a.e.), this system of ODE 's has a unique Lipschitz continuous solution for any prescribed initial vector. In this way one gets a Lipschitz continuous family of orthogonal linear maps

$$\mathbf{P}_{s_0,s_1} : T_{\gamma(s_1)}M \to T_{\gamma(s_0)}M$$

which performs the parallel transportation.

If γ is a smooth curve and if $\alpha_\epsilon(\gamma) \le \alpha$, then for each s_0 the unit vectors $\{\mathbf{P}_{s_0,s}(\dot\gamma(s)) : |s - s_0| \le \epsilon/2\}$ will lie in a cone in $T_{\gamma(s_0)}(M)$ with aperture α; i.e. there's a unit vector $e \in T_{\gamma(s_0)}(M)$ for which

(2.3) $g(\dot\gamma(s), \mathbf{P}_{s,s_0}(e)) \geq \cos(\alpha/2)$

holds whenever $|s - s_0| \leq \epsilon/2$. Note that $g(\dot\gamma(s), \mathbf{P}_{s,s_0}(e)) = g(\mathbf{P}_{s_0,s}(\dot\gamma(s)), e))$ holds, by orthogonality of $\mathbf{P}_{s_0,s_1}$.) Given $\epsilon > 0$, $\alpha_\epsilon(\gamma)$ is the smallest α with this property.

If γ is an absolutely continuous curve, then we define $\alpha_\epsilon(\gamma)$ to be the minimal α such that, for any s_0 there is an $e \in T_{\gamma(s_0)}(M)$ for which (2.3) holds for *almost* any s with $|s_0 - s| \leq \epsilon/2$.

The curve γ will be C^1 iff (2.1) holds. It will be locally Lipschitz, i.e. locally the graph of a Lipschitz continuous function $y = u(x)$ iff $\alpha_\epsilon(\gamma) < \pi$ for some small enough $\epsilon > 0$. Indeed, one easily proves the following Lemma.

Lemma 2.1. *Let γ be a unit speed parametrisation of a locally Lipschitz curve, and let a point $p = \gamma(s_0)$ on the curve be given. Assume $\epsilon > 0$ is so small that*

$$\alpha = \alpha_\epsilon(\gamma) < \pi.$$

Then there is a $\rho > 0$, which only depends on α, ϵ and R^*, such that $\gamma([s_0-\rho, s_0+\rho])$ is contained in the disk $D_\rho(p)$, and in Riemann Normal Coordinates is given as the graph $y = u(x)$ of a Lipschitz function with

$$|u'(x)| \leq 2\tan(\frac{\alpha}{2}).$$

One could actually replace the bound $2\tan(\alpha/2)$ by $(1+\eta)\tan(\alpha/2)$, for any $\eta > 0$, if one chooses ρ small enough (depending on η).

Proof. Since γ is a unit speed parametrisation, $\gamma([s_0-\rho, s_0+\rho])$ is trivially contained in $D_\rho(p)$. Choose a unit vector $e \in T_{\gamma(s_0)}(M)$ for which (2.3) holds, and let (x, y) be Riemann Normal Coordinates around p with the x axis in the direction of e. Then, since the metric is locally Euclidean, there is a $\rho > 0$ such that any tangent $\dot\gamma(s)$ with $|s - s_0| \leq \epsilon/2$ and $\gamma(s) \in D_\rho(p)$ is of the form $a\partial_x + b\partial_y$, where $a > 0$ and $|b/a| = 2\tan(\alpha/2)$. Therefore $\gamma([s_0-\rho, s_0+\rho])$ is a Lipschitz graph $y = u(x)$, with $|u'(x)| = |b/a| \leq 2\tan(\alpha/2)$. □

By smoothly patching a finite number of such Riemann Normal Coordinate neighbourhoods together , one sees that for any locally Lipschitz curve there exist an immersion $\sigma : S^1 \times [-1, 1] \to M$ and a Lipschitz function $u \in W^1_\infty(S^1)$ ($|u(x)| < 1$), such that $x \to \sigma(x, u(x))$ is a parametrization of γ.

We shall write $Lip\Omega(M)$ for the set of locally Lipschitz curves.

The next theorem is the main result of this section. It gives an a priori estimate for the "local Lipschitzness" of a solution of (0.1) in terms of $\alpha_\epsilon(\gamma_0)$.

Theorem 2.2 *Let V satisfy V_1, V_2, V_3, and let $\gamma(t)$ $(0<t<t_0)$ be a classical solution of (0.1), with locally Lipschitz initial curve $\gamma_0 \in Lip\Omega(M)$. If $\alpha_\epsilon(\gamma_0) < \pi$, then there exist $t_0 > 0$, $\bar\alpha < \pi$ and $\epsilon' < \epsilon$, which only depend on λ, μ, R^* and $\alpha_\epsilon(\gamma_0)$, such that*

$$\alpha_{\epsilon'}(\gamma(t)) \leq \bar\alpha$$

for $0 \leq t \leq t_0$.

In the proof we shall compare the $\gamma(t)$ with various curve segments which are stationary for the equation $v^\perp = V(t, k)$. We begin by defining them.

Stationary curve segments. If $\omega : (a,b) \subset \mathbf{R} \to M$ is a regular curve in M for which $V(t_\omega, k_\omega) \equiv 0$ holds, then we call ω a stationary curve segment.

Since $\lambda \leq V_k \leq \lambda^{-1}$, and $|V(t, 0)| \leq \mu$, the equation $V(t, k) = 0$ has a unique solution $k = K(t)$, for each $t \in S^1(M)$, and this solution satisfies $|K(t)| \leq \mu/\lambda$. Since V is locally Lipschitz and V_k is bounded from below, we know that $K(t)$ is a locally Lipschitz function on $S^1(M)$.

The stationary curve segments are exactly the curves in M which satisfy $k_\omega = K(\dot{\omega})$, so that their lifts to the unit tangent bundle are the trajectories of the vectorfield $X_K = t \oplus K(t)n$ on $S^1(M)$. This is a bounded vectorfield, so that it generates a complete flow on $S^1(M)$, and we may conclude the existence of a unique stationary curve segment $\omega_{p,t}$ through each point $p \in M$, in any specified direction $t \in T_pM$.

Proof of theorem 2.2. Since our initial curve is locally Lipschitz, there is a smooth immersion $\sigma : S^1 \times [-1, 1] \to M$, which allows us to parametrise γ_0 by $x \to \sigma(x, u(x))$, where u is some Lipschitz continuous function on S^1, satisfying

$$|u(x)| \le \tfrac{1}{2}, \qquad |u'(x)| \le m \text{ (a.e.)}$$

for some finite constant m. Moreover, σ can be chosen so that its derivatives of order $k \ge 1$ are bounded by some constant which only depends on k, λ, μ, R^* and $\alpha_\epsilon(\gamma_0)$.

By continuity of $\gamma(t)$ there also is a $t_1 > 0$ such that the curves $\gamma(t)$ are images under σ of regular curves $\sigma^*\gamma(t)$ when $t \le t_1$. For even smaller $t \ge 0$, these curves will be graphs $y = u(t, x)$, where $u(t, x)$ is a solution of (3.2), for the appropriate F.

For small $\delta > 0$ we define

$$A_\delta^2 = \{(x, y) \in S^1 \times [-1, 1] : |y - u(x)| \le \delta\}.$$

By theorem (5.1) there is a $t_0(\delta) > 0$ such that $\sigma^*\gamma(t)$ is contained in $A_{\delta/2}^2$ for $0 \le t \le t_0(\delta)$.

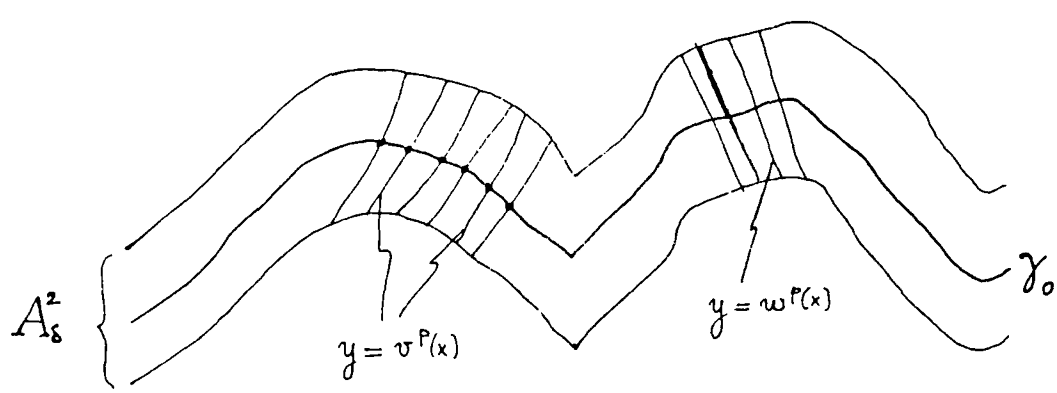

Figure 2.1— The set A_δ^2 and the graphs of v^P and w^P.

For any given point $p \in A_\delta^2$ we consider the stationary curve segment starting at p, in the direction of the vector $\partial_x + 3m\,\partial_y$. In particular, we consider the connected part that contains p and is contained in A_δ^2. Since the geodesic curvature is uniformly bounded we can choose δ so small that, for any $p \in A_\delta^2$, this segment is the graph of a function

$$v^P : (a^P, b^P) \to \mathbf{R}$$

where $v^P(a^P) = u(a^P) - \delta$, $v^P(b^P) = u(b^P) \dots$, and, more to the point, the derivative of v^P is bounded by

$$2m \le v_x^P \le 4m.$$

Similarly, the part of the stationary segment through p, in the direction $\partial_x - 3m\,\partial_y$, which lies in A_δ^2, will be the graph of a function $w^p : (c^p, d^p) \to \mathbf{R}$ whose slope lies between $-4m$ and $-2m$.

We claim that with this choice of δ, the curve $\sigma^*\gamma(t)$ remains a graph $y = u(t, x)$ with $|u_x| \le 3m$ for $t \le t_0(\delta)$. In order to reach a contradiction, we suppose that for some $t = t_*$ the curve $\sigma^*\gamma(t)$ is still the graph of $y = u(t, x)$, but contains a point p at which $u_x > 3m$. Consider the function $w(t, x) = u(t, x) - v^p(x)$ on $[a^p, b^p] \times [0, t_*]$. At $t = 0$ it has a unique zero, which is simple; this follows from $|u'| \le m$ and $v_x^p \ge 2m$. At $t = t_*$ it must have at least three sign changes, as one sees from $u_x > 3m > 2m = v_x{}^p$ at p, and the signs of $w(t_*, \cdot)$ at a^p and b^p. On the other hand, it satisfies a linear parabolic equation of the type (1.1). So, if V were $C^{2,1}$, we could apply proposition 1.2 to reach a contradiction: the number of zeroes of w cannot increase with time. In the general case, in which V is merely locally Lipschitz, w still satisfies an equation like (1.1), but the coefficients don't have to be smooth anymore. However, the number of sign changes of w still can't increase, e.g. by Matano's version [Ma] of proposition 1.2 (this version allows rougher coefficients than proposition 1.2, but it deals with sign changes instead of zeroes.) Thus u_x remains bounded from above by $3m$. By comparing with w^p one shows that u_x remains bounded from below, by $-3m$.

So, the curve $\gamma(t)$ remains locally Lipschitz for $t \in (0, t_0(\delta))$, which implies the theorem. $\square$

The same proof also applies in the following situation. Let $\gamma(t)$ be a family of evolving curve segments, which has normal velocity $v^\perp = V(t_{\gamma(t)}, k_{\gamma(t)})$, and whose end points are time independent. If the initial curve $\gamma_0 = \gamma(0)$ is locally Lipschitz, then one has:

Variation on theorem 2.2 *If $\alpha_\epsilon(\gamma_0) < \pi$, then there exist $t_0 > 0$, $\bar\alpha < \pi$ and $\epsilon' < \epsilon$, which only depend on λ, μ, R^* and $\alpha_\epsilon(\gamma_0)$, such that*

$$\alpha_{\epsilon'}(\gamma(t)) \le \bar\alpha$$

for $0 \le t \le t_0$.

3. *Local pointwise curvature estimates.*

Assuming the condition V_5^* we shall derive a local form of the smoothing property of section I.7 (i.e. of theorem I.7.4). In addition to being local, this result is also stronger in the sense that it derives boundedness of the curvature k from a given bound on $\alpha_\epsilon(\gamma(t))$, a quantity which can be defined without using the curvature of the curve. In the parabolic PDE jargon, the result gives interior L_∞ estimates of u_{xx} from L_∞ estimates of u_x.

The precise result is this.

Theorem 3.1. *Let S be the closed strip* $[-2, 2] \times \mathbf{R}$, *equipped with some smooth metric*

$$g = A(x, y)(dx)^2 + 2B(x, y)\,dx\,dy + C(x, y)(dy)^2,$$

where A, B, C, their first and second derivatives and $(AC - B^2)^{-1}$ *are uniformly bounded.*

Let $V : S^1(S) \times \mathbf{R} \to \mathbf{R}$ *satisfy* V_1, V_2, V_3 *and* V_5^*.

Assume that $\gamma(t)$ $(0 < t < 1)$ *is a classical solution of (0.1), i.e. of* $v^\perp = V(t, k)$, *which at any instant in time is the graph of a function* $y = u(t, x)$ $(|x| < 2)$ *with*

$$|u_x| \le m, \qquad (0 < t < 1,\ |x| < 2)$$

for some finite constant m.

Then there is a constant c, which only depends on λ, μ, ν, m, *the* C^2 *norms of A, B, C and the* L_∞ *norm of* $(B^2 - AC)^{-1}$, *such that*

$$|u_{xx}| \le \frac{c}{\sqrt{t}} \qquad (0 < t < 1, \ |x| < 1).$$

The hypotheses concerning A, B, C and $(AC-B^2)^{-1}$ imply that the curvature of the metric g is uniformly bounded, so that for this particular manifold (S, g) the constant R^* can be expressed in terms of the C^2 and L_∞ norms of A, B, C and $(AC-B^2)^{-1}$, respectively.

Naturally, we only need the bounds on the coefficients of the metric in the region in which the curve $\gamma(t)$ moves, rather than the whole strip S. Thus, if it were known that the curve always stays in the rectangle $[-2,2]\times[-\bar{u},\bar{u}]$, and if the metric were only defined on this rectangle, then one could extend the metric outside the rectangle in such a way that it automatically satisfies the hypotheses of the theorem.

This theorem, together with Theorem 2.1, implies short term existence for the initial value problem (0.1) with locally Lipschitz initial curves.

Theorem 3.2 *Let $V: S^1(M)\times\mathbf{R}\to\mathbf{R}$ satisfy V_1, V_2, V_3 and V_5^*.*

Then, for any initial curve $\gamma_0 \in Lip\,(\Omega(M))$ there is a maximal solution $\gamma: (0, t_{Max})\to\Omega(M)$, such that $\gamma(t)\to\gamma_0$ as $t\to 0$.

More precisely, if $\sigma: S^1\times[-1,1]\to M$ is an immersion for which $\sigma^*(\gamma_0)$ is the graph of a Lipschitz function $y=u_0(x)$, then for small $t>0$ the curve $\sigma^*(\gamma(t))$ is the graph of another Lipschitz function $y=u(t,x)$, and $u(t,x)\to u_0(x)$ uniformly as $t\to 0$.

Proof. Approximate u_0 by smooth functions u_n, whose first derivatives are uniformly bounded. For each of these functions one can solve the initial value problem on some time interval $[0, t_n)$. From the proof of theorem 2.1 one sees that there is a $t_* > 0$, such that, as long as $t < t_*$, the solutions $\gamma_n(t)$ remain images under σ of graphs of uniformly Lipschitz continuous functions $u_n(t,\cdot)$. Therefore, by theorem I.9.1, the solutions cannot blowup before t_*, i.e. for all n we have $t_n \ge t_*$. We may assume that $t_* < 1$, so that theorem I.3.1 gives us a uniform upper bound for the curvature of the $\gamma_n(t)$ (namely $ct^{-1/2}$), and hence we know that the curvatures are uniformly Hölder continuous, provided t stays away from zero. All this allows us to extract a convergent subsequence of the $\gamma_n(t)$ whose limit $\gamma(t)$ is the solution we are looking for. Indeed, it is a classical solution, since its curvature is bounded by $ct^{-1/2}$ for $t < \min(1, t_*)$, and arguments similar to those in section eight show that the limit solution has γ_0 as initial value. □

Before we prove the theorem, we observe that boundedness of u_x implies Hölder continuity of u in time. The analogous statement for quasilinear equations is well known; in our present situation it is a direct consequence of the large scale displacement estimate of section I.5.

Lemma 3.3. *Let u be as in theorem 3.1. Then u is Hölder continuous in time for all $|x| \le 1$, i.e. there's a constant c such that*

$$|u(t_0,x) - u(t_1,x)| \le c\,|t_1 - t_0|^{1/2}.$$

The constant only depends on λ, μ, m and the C^2 norms of A, B, C and the L_∞ norm of $(AC-B^2)^{-1}$.

Proof of the lemma. We may assume that $t_0 < t_1 < t_0 + t_*$, where t_* is given in theorem I.5.1. The arguments in the proof of that theorem show that

$$\gamma(t_1) \subset \mathbf{N}_{c\sqrt{t_1-t_0}}(\gamma(t_0)\cup\partial S),$$

where distances are to be measured with the metric g. By choosing t_* smaller, if necessary, we can ensure that $\mathbf{N}_{c\sqrt{t_*}}(\partial S)$ is disjoint from the strip $[-1,1]\times\mathbf{R}$. Then, for any $t_0 < t_1 < t_0 + t_*$ one has

$$\gamma(t_1)\cap[-1,1]\times\mathbf{R} \subset \mathbf{N}_{c\sqrt{t_1-t_0}}(\gamma(t_0)).$$

It is geometrically obvious that the condition $|u_x| \le m$ implies that the Euclidean distance of a point $(x_0, u_0) \in S$ to the curve $\gamma(t)$ is bounded from above and from below by a multiple of $|u_0 - u(t, x_0)|$. We have assumed that A, B, C, and $(AC - B^2)^{-1}$ are uniformly bounded, so that distances measured in the metric g, and distances measured in the Euclidean metric are proportional by a factor which is both bounded from above and below. Hence one obtains the inequality of the lemma. □

The proof shows that u will be Hölder continuous in time on any set $|x| \le \rho$, as long as $\rho < 2$. Of course, one will have to choose the constant $c = c_\rho$ larger and larger as ρ gets close to 2.

Proof Theorem 3.1. We begin with a rescaling argument which shows that we only have to bound u_{xx} at $t=1$. Let $\sigma \in (0, 1)$ be given, and consider $\hat{u}(t, x) = \sigma^{-1} u(\sigma^2 t, \sigma x)$. The graph of $\hat{u}$ satisfies (0.1) with a new V, $\hat{V} = \sigma V(t, \sigma^{-1} k)$, which also satisfies $V_1 \cdots V_5^*$, with the same constants λ, μ, ν. The geodesic curvature k should be measured with the rescaled metric $\sigma^{-2} g$, which, in the rescaled coordinates $(\xi = x/\sigma, \eta = y/\sigma)$, may be written as $A_\sigma (d\xi)^2 + 2B_\sigma d\xi d\eta + C_\sigma (d\eta)^2$. The C^2 norms of $A_\sigma, B_\sigma, C_\sigma$ and the L_∞ norm of $(A_\sigma C_\sigma - B_\sigma^2)^{-1}$ do not exceed the corresponding C^2 norms of A, B, C and the L_∞ norm of $(AC - B^2)^{-1}$, as one sees after a short computation. Finally, the rescaled u is also Lipschitz in x, and has the same Lipschitz constant, m, as the original u.

If we had a bound $|u_{xx}| < c$ at $t=1$, then this bound would also apply to $\hat{u}$. But this would imply $|u_{xx}| < c\sigma^{-1}$ at time $t = \sigma^2$, which is what we want. So, from here on, we shall concentrate on the estimate for u_{xx} at $t=1$.

As before, the equation $v^\perp = V(t, k)$ is equivalent to the parabolic p.d.e.

(I.3.1) $\quad u_t = F(x, u, u_x, u_{xx})$

for the function $u(t, x)$. The lemma tells us that u is uniformly bounded for $|x| < 3/2$ and $t < 1$, so that, by increasing m, if necessary, we may assume that $|u| \le m$. Since $|u_x| \le m$, we may modify $F(x, u, p, q)$ in the region $|p| \ge m$ so that it satisfies $F_1 \cdots F_4$ of section I.10. In particular, we can assume that $F(x, u, p, q) \equiv q$ when $|p| \ge 2m$ or $|u| \ge 2m$ holds. Our modified F can therefore be written as

$$F(x, u, p, q) = A(x, u, p, q)q + B(x, u, p)$$

where $\lambda \le A \le \lambda^{-1}$, and $B = F(x, u, p, 0)$ vanishes for $|u| \ge 2m$, and is uniformly bounded by $|B| \le B^*$ otherwise.

Consider

$$v(t, x) = -3m + ae^{-\alpha t} \cos\frac{x - \xi}{h} \qquad (|x - \xi| \le \frac{\pi h}{2}, t \ge 0).$$

Then

$$v_t = -\alpha(v + 3m), \text{ and } v_{xx} = -h^{-2}(v + 3m),$$

and, using $|B| \le \frac{B^*}{m}(v + 3m)$, you find:

$$v_t - F(x, v, v_x, v_{xx}) \ge (-\alpha + \frac{\lambda}{h^2} - \frac{B^*}{m})(v + 3m),$$

$$v_t - F(x, v, v_x, v_{xx}) \le (-\alpha + \frac{1}{\lambda h^2} + \frac{B^*}{m})(v + 3m).$$

So v is a subsolution if $v_t - F(\cdots) \le 0$, i.e., if

$$\alpha \geq \lambda^{-1} h^{-2} + \frac{B^*}{m} \equiv \bar{\alpha},$$

and v is a supersolution if $v_t - F(\cdots) \geq 0$, i.e., if

$$\alpha \leq \lambda h^{-2} - \frac{B^*}{m} \equiv \underline{\alpha},$$

For each $a > 0$ and $\xi \in [-1, 1]$ we let $w^{a,\xi}$ be the solution of the Cauchy Dirichlet problem

$$w_t = F(x, w, w_x, w_{xx}) \qquad (0 < t < 1, \ |x - \xi| < \pi h/2)$$

$$w(t, \xi \pm \pi h/2) \equiv 0 \qquad (0 < t < 1)$$

$$w(0, x) = v^{a,\xi}(0, x) = -3m + a\cos\frac{x-\xi}{h}.$$

The initial data for this problem are smooth, so a solution will certainly exist for a short time, and if it fails to exist all the way up to $t = 1$, then the second derivative of w must blow up at some $t_0 < 1$. For as long as the solution exists it satisfies the following inequalities,

$$-3m + ae^{-\bar{\alpha}t}\cos\frac{x-\xi}{h} \leq w^{a,\xi} \leq -3m + ae^{-\underline{\alpha}t}\cos\frac{x-\xi}{h}.$$

Therefore there is a neighbourhood of $\xi \pm \pi h/2$, of fixed size, on which one has $w < -2m$. We have chosen our F so that w satisfies the ordinary heat equation in this region, so that the solution cannot blow up near the boundary. The proof of theorem I.9.1 shows that the solution cannot blow up in the interior of the interval $(\xi - \pi h/2, \xi + \pi h/2)$ either. So we can conclude that the solution $w^{a,\xi}$ does indeed exist, for any $a > 0$ and $\xi \in [-1, 1]$.

The same argument also shows that, if you have a sequence of solutions, with different but bounded a's and ξ's, then their second derivatives $w_{xx}^{a,\xi}$ must remain uniformly bounded for $0 \leq t \leq 1$. Therefore they are also uniformly Hölder continuous on $\delta \leq t \leq 1$, for any $\delta > 0$, and you can extract a subsequence which converges to some classical solution of the PDE. By uniqueness of the solution $w^{a,\xi}$, this implies that the $w^{a,\xi}$ depend continuously on a and ξ.

In the next step of the proof we determine a range $a_0 \leq a \leq a_1$ and choose the constant $h > 0$ so small that

(i) *for any* $a \in [a_0, a_1]$, *one has* $|v_x^{a,\xi}(0,x)| \geq 2m$ *whenever* $|v^{a,\xi}(0,x)| \leq m$, *so that the graphs of* $v(0,x)$ *and* $u(0,x)$ *intersect exactly twice.*

(ii) *at* $t = 1$ *one has* $w^{a_0,\xi} \leq -m$ *and* $w^{a_1,\xi}(1,\xi) \geq m$, *so that* $w^{a_0,\xi}(1,\cdot)$ *and* $u(1,\cdot)$ *have disjoint graphs, and* $w^{a_1,\xi}(1,\cdot)$ *and* $u(1,\cdot)$ *still intersect twice.*

To determine a_0, a_1 we observe that

$$|v_x| = \left|-\frac{a}{h}\sin\frac{x-\xi}{h}\right| = \left[(a/h)^2 - (v+3m)^2\right]^{1/2}$$

$$\geq (a/h)^2 - (4m)^2$$

which exceeds m^2 when $a/h > m\sqrt{17}$. We shall take $a_0 = 5hm$, so that condition (i) above is satisfied.

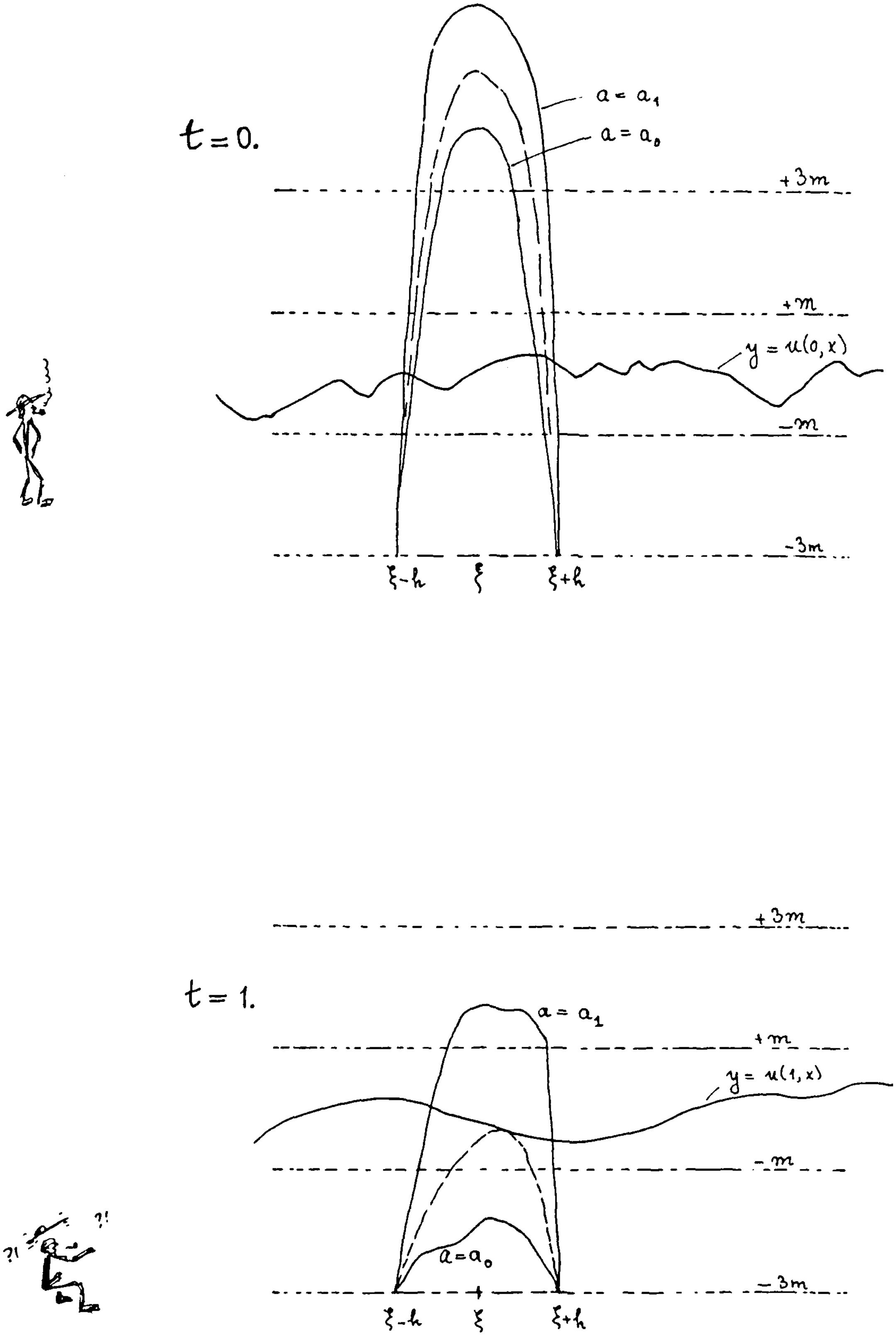

Figure 3.1 The $w^{a,\xi}$ at $t=0$ and at $t=1$.

Turning to the other condition, we note that

$$w^{a_0,\xi}(1,x) \le -3m + 5hme^{-\underline{\alpha}} = (5he^{-\underline{\alpha}} - 3)m,$$

so that we want $5he^{-\underline{\alpha}} - 3 < -1$. This will certainly hold if $h < 2/5$ and $\underline{\alpha} \ge 0$. Recalling the definition of $\underline{\alpha}$ (i.e., how it depends on h), we see that the first part of condition (ii) is satisfied, if we choose $h = \min(1/5, \sqrt{\lambda m / c})$.

Finally, one has

$$w^{a_1,\xi}(1,\xi) \ge -3m + a_1 e^{-\bar{\alpha}},$$

which will be larger than $+m$, if $a_1 > 4me^{\bar{\alpha}}$; we choose $a_1 = 5me^{\bar{\alpha}}$.

Both $u(t,x)$ and $w^{a,\xi}(t,x)$ are solutions of the PDE (I.3.1) so that their difference, $z(t,x) = u(t,x) - w^{a,\xi}(t,x)$, satisfies a linear equation like (11.1). As before, the number of sign changes of $z(t,\cdot)$ cannot increase with time. By condition (i), we know that, at $t = 0$, $z(t,\cdot)$ has exactly two zeroes, both of which are simple.

For any $t \in [0,1]$ we know that $z(t, \xi \pm \pi h/2) > 0$.

Assume that, at $t = 1$, z has a multiple zero, i.e. that $z(1,x_0) = z_x(1,x_0) = 0$, for some $x_0 \in (\xi - \pi h/2, \xi + \pi h/2)$. Then we must have $z_{xx}(1,x_0) \ge 0$; for, if $z_{xx}(1,x_0) < 0$, then $z(1,\cdot)$ would have at least three zeroes (recall that z is positive at $\xi \pm \pi h/2$). This contradicts proposition 11.1, if V is at least $C^{2,1}$. In general, it follows from $z = z_x = 0$ and $z_{xx} < 0$, that $z_t < 0$, so that for some $t_1 < 1$ close to 1 one has $z(t_1,x_0) < 0$. Thus $z(t_1,\cdot)$ must have three sign changes, which contradicts Matano's version of proposition 11.1.

This observation is of importance because it gives us an estimate of $u_{xx}(1,x_0)$. Indeed we have

$$u_{xx}(1,x_0) \ge c$$

where $c \equiv \inf w_{xx}^{a,\xi}(1,x)$; the infimum is taken over all $a \in [a_0,a_1]$, $|\xi| \le 1$ and $x \in (\xi - \pi h/2, \xi + \pi h/2)$. We have already remarked that c exists, and that it can be bounded from below by a constant which only depends on λ, μ, ν, m and R^*.

To complete the proof we have to show that for any $x_0 \in (-1,1)$ one can find a, ξ such that $u(1,\cdot) - w^{a,\xi}(1,\cdot)$ has a multiple zero at $x = x_0$. This would imply that the estimate (3.2) holds at any prescribed x_0. An upper bound for u_{xx} follows by applying the whole argument to $-u$ instead of u.

So let $x_0 \in (-1,1)$ be given. Then we introduce the map

$$\Phi : D = [a_0,a_1] \times [x_0 - \pi h/2, x_0 + \pi h/2] \to \mathbf{R}^2,$$

given by $\Phi(a,\xi) = \left(w^{a,\xi}(1,x_0) - u(1,x_0),\ w_x^{a,\xi}(1,x_0) - u_x(1,x_0) \right)$.

Let ϕ_1 and ϕ_2 be the first and second coordinates of Φ, respectively.

We have to show that $(0,0)$ lies in the range of Φ; arguing by contradiction, we assume this is not the case.

The solution $w^{a,\xi}$ depends continuously on the parameters a and ξ, so that the map Φ is continuous. We shall compute its winding number on ∂D.

On the three portions $\{a_0\} \times [x_0 - \pi h/2, x_0 + \pi h/2]$, and $[a_0, a_1] \times \{x_0 \pm \pi h/2\}$ one has $w < u$, so that Φ maps these sides into the left half plane ($\phi_1 < 0$).

On the remaining side, where $a = a_1$, we have

$$w^{a_1,\xi}(1,x) \geq -3m + a_1 e^{-\bar{a}} \cos\frac{x-\xi}{h}.$$

If $\xi > x_0$, and $\phi_1(a_1, \xi) = 0$, then $\phi_2(a_1, \xi) > 0$. Indeed, by assumption we cannot have equality. If the opposite inequality were to hold, then the difference $w^{a_1,\xi} - u$, as a function of x, would have to have at least four zeroes in the interval $(\xi - \pi h/2, \xi + \pi h/2)$.

So Φ maps $\{a_1\}\times[0, x_0+\pi h/2]$ into the plane with the negative y axis removed, i.e. $\mathbf{R}^2 - \{0\}\times(-\infty, 0]$. Similarly Φ maps the other half of the fourth side of D into the plane with the positive y axis removed. Our condition (ii) implies that $\Phi(a_1, x_0)$ belongs to the right half plane, so that we can conclude that $\Phi(\partial D)$ loops around the origin once, and therefore also that the origin lies in the range of Φ. This contradicts our initial assumption and the proof is complete. □

4. The initial value problem for unoriented curves.

In this section we shall assume that V satisfies the conditions V_1, V_2, V_3 and V_5^*, as well as the symmetry condition S.

Theorem 4.1. *For any C^1 graphlike initial curve γ_0, there exists a classical solution $\gamma : (0, t_0) \to \Omega(M)$ of (0.1), with γ_0 as initial value.*

Proof. Let $\sigma : S^1 \times [-1, 1] \to M$ be the local homeomorphism and let $u \in C^0(S^1)$, be the continuous function with $|u(x)| \leq 1/2$ which have properties (1,2,3) that were mentioned in the introduction.

Let $\sigma_{\xi 0}$ be the curve segment $\{\xi\}\times[-1, 1]$ (with $\xi \in S^1$), and let $\rho_{\xi 0}$ be the graph of

$$x = y + \varsigma(y) \quad (mod \mathbf{Z}) \qquad (-1 \leq y \leq 1),$$

where ς is a smooth, even function which satisfies $\varsigma(y) \equiv 0$ for $|y| \leq 3/4$, and $\varsigma(y) > 0$ for $3/4 < y \leq 1$ (see fig 4.1). We define $h = \varsigma(1)$.

For some small $t_0 > 0$ we let $\rho_\xi(t)$ and $\sigma_\xi(t)$ $(0 \leq t \leq t_0)$ be smooth curve segments whose images under σ evolve according to (0.1), whose end points remain fixed, and which also depend continuously, in the C^1 topology, on ξ and t. At t=0, the curve segments should coincide with $\rho_{\xi 0}$ and $\sigma_{\xi 0}$ respectively.

Such curve families can be constructed, after one modifies the function V near the edges of the immersion σ, i.e. near $\sigma(S^1\times\{1, -1\})$. This modification may be necessary, since we don't have an existence theorem for the initial value problem for curves *with boundary* which evolve with $v^\perp = V(t, k)$. For smooth initial data one can repeat the procedure in section three of part I, and use the theory in [Ei] or [DPG, A1] to obtain a local solution. Just as in the case of closed curves, the maximal solution exists until the L_∞ norm of its curvature blows up. Now we have a local Lipschitz bound for evolving curve segments (the "...iation on theorem 2.2"), and the main result of the previous section therefore gives us a bound for the curvature, away from the end points of the curve segment $\gamma(t)$. So, if the curvature of $\gamma(t)$ blows up, then it must do so at one of its end points. To prevent this from happening, we change V in a very small neighborhood U of $\sigma(S^1\times\{-1, 1\})$, so that $V^{new}(t, k) \equiv k$ for all k and t whose base point lies in a smaller neighborhood $U' \subset U$ of $\sigma(S^1\times\{-1, 1\})$. This has the following effect. If you represent the $\sigma_\xi(t)$ and $\rho_\xi(t)$ as graphs of functions, then near their end points the equation $v^\perp = V(t, k)$ reduces to a *quasilinear* parabolic equation, instead of a fully nonlinear one. For quasilinear equations one can appeal to the classical results of Ladyzhenskaya, Ural'ceva and Solonnikov ([LUS]), which imply that the local Lipschitz bound forces the curvature to remain bounded near the end points.

So, if we let the $\sigma_\xi(t)$ and $\rho_\xi(t)$ evolve with V^{new} instead of V, then we have L_∞ bounds for their curvatures, as long as $\alpha_\epsilon(\sigma_\xi(t)) < \pi$ and $\alpha_\epsilon(\rho_\xi(t)) < \pi$, for some $\epsilon > 0$. In particular, the families $\rho_\xi(t)$ and $\sigma_\xi(t)$ will exist for a short time $t_0 > 0$, which is independent of ξ.

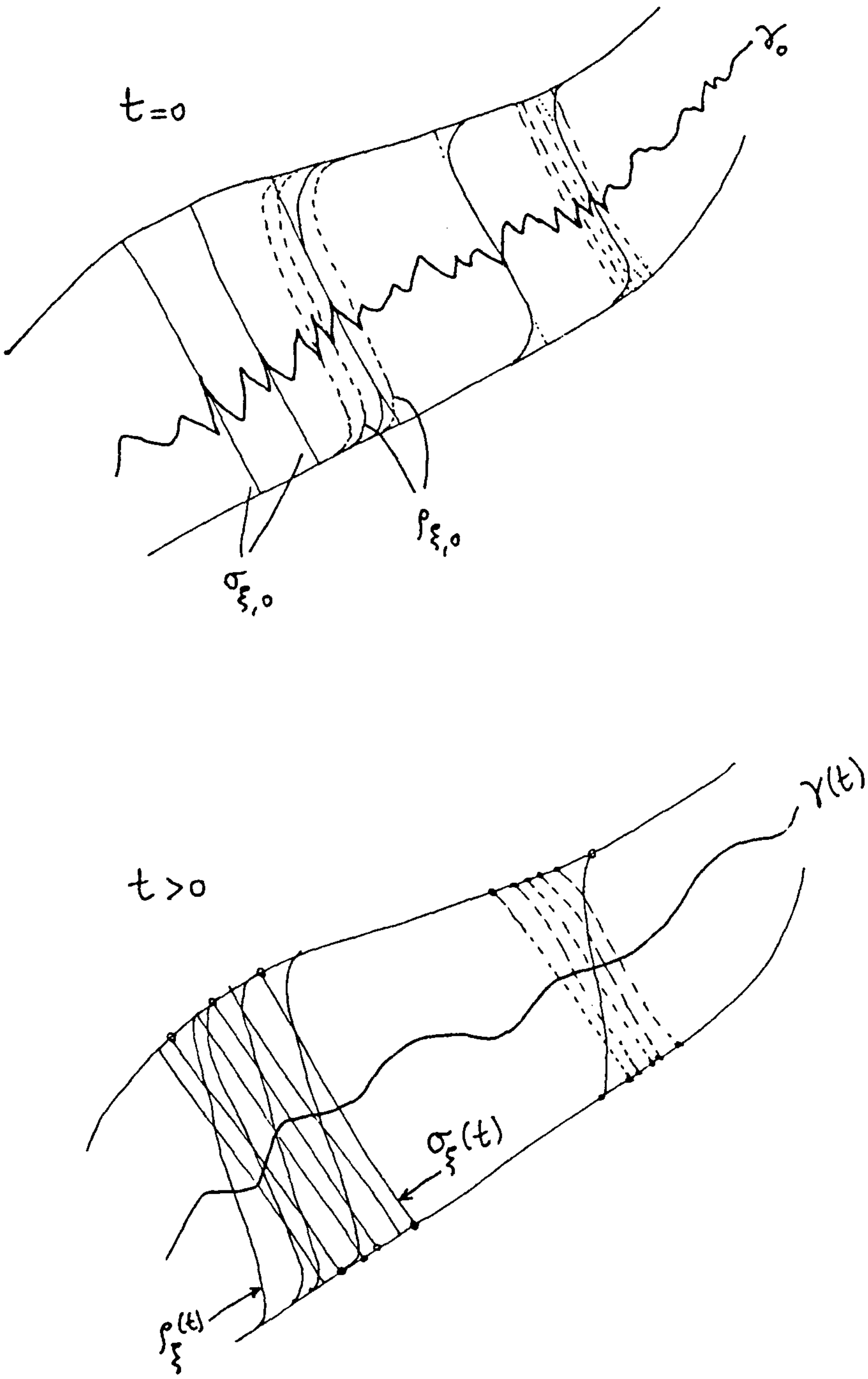

Figure 4.1 The σ and ρ foliations at $t=0$ and at some $t>0$.

At each $t>0$ the curves $\sigma_\xi(t)$ form a continuous foliation of the annulus $S^1\times[-1,1]$, as do the curves $\rho_\xi(t)$. At each $t>0$ these foliations will be transversal, and the minimal angle at which the curves $\rho_{\xi_0}(t)$ and $\sigma_{\xi_1}(t)$ will intersect is bounded from below.

Now choose C^1 functions $u_n(x)$ which converge uniformly to $u(x)$, and let $u_n(t,x)$ be such that the images under σ of the graphs of $u_n(t,\cdot)$ evolve according to (0.1), and such that $u_n(0,x)\equiv u_n$. By our existence theorem in section 3, the u_n will be defined on some time interval $[0,t_n]$.

At $t=0$ each u_n will be transversal to both the $\sigma_{\xi 0}$ and the $\rho_{\xi 0}$ foliations, and because our V^{new} satisfies the symmetry condition (S), it follows as in section 1, that the graph of $u_n(t,\cdot)$ will be transversal to the $\sigma_\xi(t)$ and $\rho_\xi(t)$ foliations. Since these foliations intersect each other at a nonzero angle, this implies that the $u_n(t,\cdot)$ are Lipschitz curves, and uniformly so in $n\in\mathbb{N}$. But this implies that the $u_n(t,\cdot)$ cannot blow up before $t=t_0$, and that they have uniformly bounded curvatures on any time interval $[\delta,t_0]$ $(\delta>0)$. Some subsequence of the u_n's will therefore converge to a classical solution of (0.1), and using theorem I.5.1 one verifies that this limiting classical solution has the graph of $u(x)$ as its initial value.

Thus we have a local solution of (0.1) for the modified V^{new}, and since V and V^{new} coincide in a neighbourhood of γ_0, there's an initial segment $\{\gamma(t),0\le t<t_1\}$ which is a solution of (0.1) for the original V. □

5. The limit curve in the unoriented case.

Let $\gamma:(0,t_{Max})\to\Omega(M)$ be a classical solution of (0.1). Then we have shown in section five, that the sets $\gamma(t)$ converge in the Hausdorff metric to a limit set γ^* as $t\to t_{Max}$. In this section we shall show that this limit curve actually is piecewise smooth.

Theorem 5.1. *Assume that V satisfies V_2, V_3, V_5^* and S, and let V be at least $C^{2,1}$.*

If $\gamma:[0,t_{Max})\to\Omega(M)$ is a maximal classical solution with $t_{Max}<\infty$, then there exist a finite number of points $\{Q_1,\cdots,Q_m\}$ on the limit curve γ^*, such that γ^*-$\{Q_1,\cdots,Q_m\}$ consists of smooth ($C^{m+2,\alpha}$) arcs. Away from the points Q_j the curve $\gamma(t)$ converges in the C^{m+2} topology to γ^*.

The number of singular points does not exceed $K(t_{Max})/\pi$, where $K(t_{Max})$ is the upper bound for the total absolute curvature of γ^*.

Proof. We may assume that the length of $\gamma(t)$ does not tend to zero as $t\to t_{Max}$, since the theorem is trivially true in this case.

For each $t<t_{Max}$ we let $\gamma_t:S^1\to M$ be a constant speed parametrisation of $\gamma(t)$. On S^1 we have the measure $|k(t,s)|\,ds$, and we define $\mathbf{K}_t$ to be the pushforward of this measure under the map $\gamma_t:S^1\to M$; thus $\mathbf{K}_t$ is a Borel measure on M, and by theorem I.4.1 the measures $\mathbf{K}_t$ are uniformly bounded for $t<t_{Max}$.

We select a sequence $t_n\to t_{Max}$ for which the measures $\mathbf{K}_{t_n}$ converge weakly to some limit measure $\mathbf{K}$. This limit measure can be decomposed into its atoms and a continuous part, i.e. we can write $\mathbf{K}$ as

$$\mathbf{K}=\mathbf{K}_c+\sum_i \kappa_i\delta_{Q_i},$$

where $\mathbf{K}_c(\{P\})=0$ for any point $P\in M$, $\{Q_i\}$ is an at most countably infinite sequence of points in M and $\kappa_1\ge\kappa_2\ge\kappa_3\ge\cdots>0$ is a sequence with $\sum\kappa_i<\infty$. Let m be the smallest integer for which $\kappa_{m+1}<\pi$; by theorem I.9.1 m is not less than 1, i.e. one of the κ's must be $\ge\pi$.

Let P be any point in M-$\{Q_1,\cdots,Q_m\}$ which lies on the limit curve γ^*. Then there is an $\epsilon>0$ such that $\mathbf{K}(B_\epsilon(P))<\pi$, and after throwing away the first few t_n's, if necessary, we may assume that $\mathbf{K}_{t_n}(B_\epsilon(P))\le\pi-\alpha$ for all n and some $\alpha>0$.

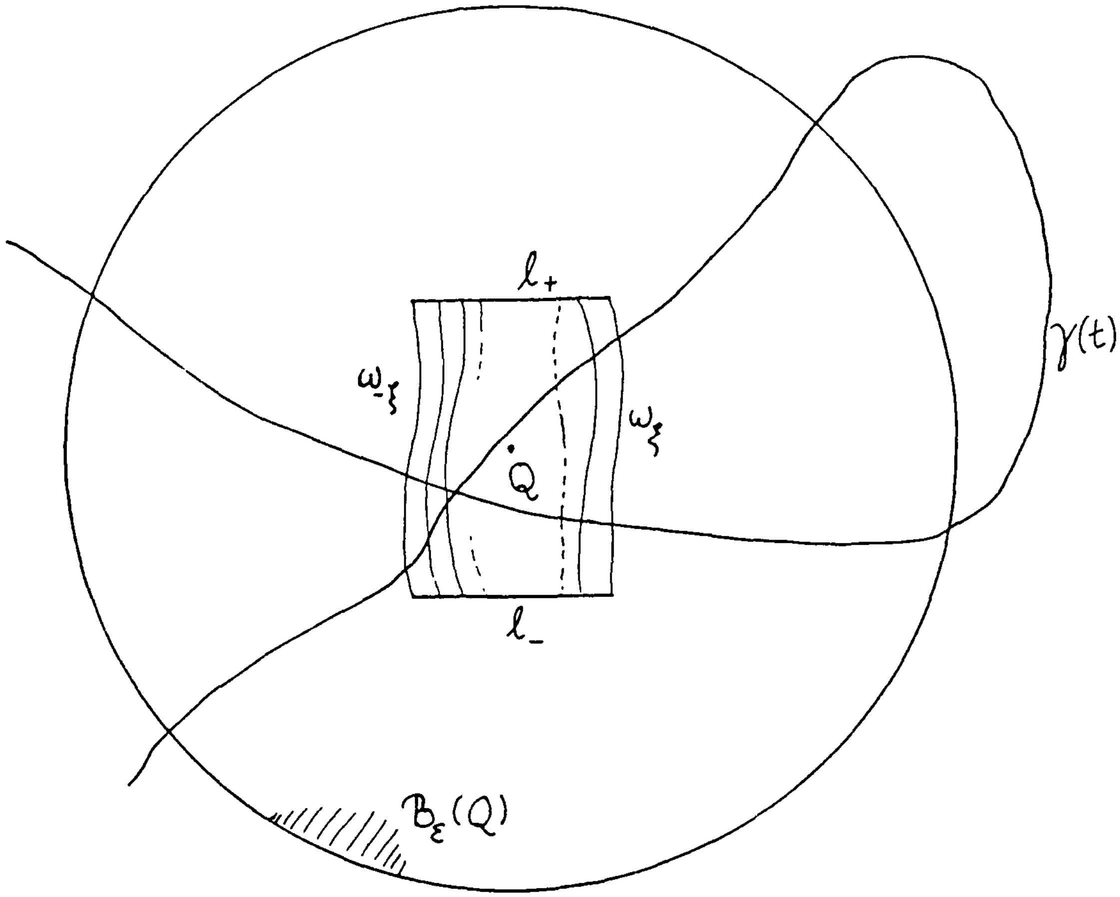

Figure 5.1 The region R, and the ω-foliation.

For any n, the preimage of $B_\epsilon(P)$ under γ_{t_n} will consist of a countable family of intervals $J_i \subset S^1$. Assuming that ϵ is so small that none of the $\gamma(t_n)$'s is contained in $B_\epsilon(P)$, the length of any J_i whose image intersects $B_{\epsilon/2}(P)$ is bounded from below by ϵ/L, where L is the length of $\gamma(t_n)$. Therefore there cannot be more than $L/(\epsilon/L) = L^2\epsilon^{-1}$ of such intervals. By passing to a subsequence of $\{t_n\}$ we may assume that this number of intervals is constant, say k.

Choose Riemann Normal Coordinates (x, y) on $B_\epsilon(P)$. One can write the unit tangent to $\gamma(t_n)$ at $\gamma_{t_n}(s)$ as

$$t(t_n, s) = r[\cos(\theta)\partial_x + \sin(\theta)\partial_y] ,$$

where $r(t_n, \cdot)$, $\theta(t_n, \cdot)$ are continuous functions on J_i.

If the metric of the manifold M were flat on $B_\epsilon(P)$, then the condition $\mathsf{K}_{t_n}(B_\epsilon(P)) \leq \pi - \alpha$ would imply that the range of $\theta(t_n, \cdot)$ on the J_i's would consist of k intervals whose combined total length does not exceed $\pi - \alpha$. The complement of this range therefore contains an interval of length at least α/k. In general the metric will not be flat, but by choosing ϵ small enough (compared with the maximal curvature R^* of M) we may assume that the complement of the range of $\theta(t_n, \cdot)$ contains an interval of length $\alpha/2k$. This interval might depend on n, but after passing to a subsequence again, we can assume that there is an interval of length $\alpha/4k$ which lies in the complement of the ranges of all the $\theta(t_n, \cdot)$'s. After rotating the coordinates we may also assume that this interval is $[\pi/2 - \alpha/8k, \pi/2 + \alpha/8k]$. The final conclusion is that we may assume that at each t_n there are exactly k components of $\gamma(t) \cap B_\epsilon(P)$ which also intersect $B_{\epsilon/2}(P)$, and that these components are graphs of uniformly Lipschitz functions $u_{n,1}(x), \cdots, u_{n,k}(x)$. Their Lipschitz constant is in fact bounded by $\cot(\alpha/8k)$.

Since $\gamma(t)$ converges in the Hausdorff metric to γ^*, the functions $u_{n,i}(x)$ $(1 \leq i \leq k)$ will converge uniformly. Their limits will parametrise $\gamma^* \cap B_{\epsilon/2}(P)$.

Choose $\xi, \eta > 0$ so that the two line segments $l_\pm = [-\xi, \xi] \times \{\pm\eta\}$ lie inside $B_{\epsilon/2}(P)$ and are disjoint from γ^*. By throwing away a finite number of t_n's, if necessary, we can arrange that $l_\pm$ are disjoint from all the $\gamma(t_n)$ as well.

For any $x \in [-\xi, \xi]$ we let ω_x be the unique stationary curve segment in $B_{\epsilon/2}(P)$ with $(x, -\eta)$ and (x, η) as end points. As we explained in section twelve, these segments are obtained by integrating the vector field $X_K = t \oplus K(t)n$ on the unit tangent bundle $S^1(M)$. The boundedness of the function $K: S^1(M) \to \mathbf{R}$ implies that the segments ω_x do indeed exist and are uniquely determined by their end points, if ϵ is small enough. Moreover, they form a foliation of a small (nonlinear) rectangle R whose sides are $\omega_{\pm\xi}$ and $l_\pm$ (see figure 5.1.)

Now let n_0 be so large that $\gamma(t)$ and the line segments $l_\pm$ are disjoint for all $t \in (t_{n_0}, t_{Max})$ (and not just at the t_j's). At each time t_j $(j \geq n_0)$ we know that $\gamma(t)$ intersects each ω_x at most a finite number of times; by passing to a subsequence again we can ensure that this number is constant, say $k_1 \leq k$. Since the curves $\gamma(t)$ stay away from the end points of each ω_x, the arguments of section one (theorem 1.3) show that the number of intersections of $\gamma(t)$ with ω_x never increases, and drops at any time t, when $\gamma(t)$ and ω_x have a nontransversal intersection. Thus the number of intersections of $\gamma(t)$ with ω_x remains constant (k_1) after $t = t_{n_0}$, and $\gamma(t)$ must intersect the ω_x's transversally for $t \geq t_{n_0}$.

In other words, for $t \geq t_{n_0}$, the intersection $\gamma(t) \cap \mathsf{R}$ consists of k_1 curves which are graphlike with respect to the foliation $\{\omega_x\}$. Arguing as in the previous section, this implies that the k_1 curve segments which make up $\gamma(t) \cap \mathsf{R}$ remain uniformly Lipschitz, and hence that their curvatures remain uniformly bounded and Hölder continuous in the interior of R. Therefore $\gamma(t) \cap \mathsf{R}$ remains smooth, and converges to $\gamma^* \cap \mathsf{R}$, which also must be smooth. We can conclude that the limit curve is indeed smooth, except, maybe, at the points $Q_1, \cdots, Q_m$. $\square$

6. The reduced limit curve.

To get a better description of the limit curve, we choose a parametrization $\Gamma : S^1 \times [0, t_{Max}) \rightarrow M$ of the solution $\gamma : [0, t_{Max}) \rightarrow \Omega(M)$, for which $\Gamma_t(u, t) \perp \Gamma_u(u, t)$ holds for all $(u, t) \in S^1 \times [0, t_{Max})$. As in section six, we write the arclength as

$$ds = J(u, t) du,$$

where J is the length of Γ_u; J satisfies $J_t = -kv^{\perp}J$.

By theorem 6.2 the $\Gamma(u, t)$ converge (uniformly) to some Lipschitz continuous map $\Gamma_* : S^1 \rightarrow M$. In general Γ_* need not be a local homeomorphism. In what follows, we try to see "how bad" Γ_* can be, by looking at its self intersections.

Let $U \subset S^1$ be the set of u's for which $\Gamma_*(u)$ is not one of the singular points Q_j. Since Γ_* is continuous the set U is open in S^1.

We claim that U consists of a finite number of intervals. If this were not true, then U would be the union of a countable family of disjoint intervals $U = \cup_{i=1}^{\infty} U_i$. On each of these intervals Γ_* parametrizes a curve segment going from one of the singular points to another (or, possibly, the same). Since the their total length is finite, all but a finite number of these will begin and end at the same point. Thus all but a finite number of these intervals get mapped into arbitrary small neighbourhoods of the Q_i, which implies that all but a finite number of them will have total absolute curvature $\geq \pi - \epsilon$, for some $\epsilon > 0$. Since the total absolute curvature of the limit curve is finite, this is a contradiction.

Finiteness of the total absolute curvature also implies that Γ_* is actually C^1 on $\bar{U}$, and that the unit tangent

$$t(u, t) = \frac{\Gamma_u(u, t)}{J(u, t)} \in T_{\Gamma(u, t)}(M)$$

is continuous on $\bar{U}$.

Write $U = \cup_{i=1}^{k} U_i$, with U_i disjoint intervals, and $S^1 - U = \cup_{i=1}^{k} I_i$, where the I_i are closed intervals (or points). For small positive δ, let $I_{i, \delta}$ denote the δ neighbourhood of I_i; we can choose δ so small that the $I_{i, \delta}$ are disjoint. Also define

$$\Phi(t) = \int_{I_{i, \delta}} \phi(k(u, t)) J(u, t)\, du$$

where $\phi(k) = |k|$ if $|k| \geq 1$, and $\phi(k) = 1 - k^2/2$ otherwise. Then, just as in section four of part I, one shows that $\Phi'(t)$ is bounded from above, so that $\Phi(t)$ is of bounded variation, and $\lim_{t \rightarrow t_{Max}} \Phi(t)$ exists. Indeed, the only change we have to make in the arguments in section four, concerns the boundary terms which appear when we integrate by parts in (I.4.4). These are certainly bounded, since our solution stays smooth on $\partial I_{i, \delta}$. Thus

$$K_{i, \delta}(t) = \int_{I_{i, \delta}} |k(u, t)| J(u, t)\, du$$

also converges to some $K_{i, \delta}(t_{Max})$, as $t \rightarrow t_{Max}$.

The same blow up argument which was used to prove theorem 9.1 also implies:

Lemma 6.1. *Either* $K_{i, \delta}(t_{Max}) \geq \pi$ *for all* $\delta > 0$, *or* $\lim_{\delta \rightarrow 0} K_{i, \delta}(t_{Max}) = 0$.

Proof. If the curvature of $\Gamma(t,\cdot)$ remains uniformly bounded in $I_{i,\delta}$ as t tends to t_{Max} then the limit Γ_* is smooth with nonvanishing derivative on $I_{i,\delta}$. The interval $I_{i,\delta}$ therefore must be a point $\{u_0\}$, and one has $|K_{i,\delta}| \leq C\delta$ for some $C>0$. If this happens, then $\Gamma_* \mid I_{i,\delta}$ is a regular part of the limit curve, which just happens to pass through one of the Q_i's at the time that another part of the curve $\gamma(t)$ became singular there.

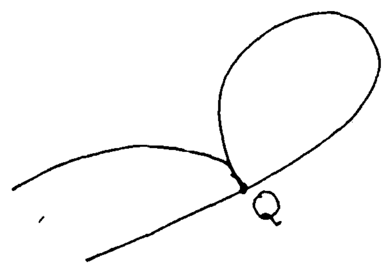

Figure 6.1— A regular part of Γ_* passes through one of its singularities.

If the curvature of $\Gamma(t,\cdot)$ becomes unbounded in $I_{i,\delta}$ then we can select a convergent sequence (u_n, t_n) along which k blows up. Since $\Gamma(t,\cdot)$ remains smooth outside of I_i, the limit of the u_n's must lie in $I_{i,\delta}$, so that the arguments in section I.9 go through without any change. The conclusion is that the total absolute curvature of Γ_* in $I_{i,\delta}$ must be at least π. $\square$

Next, we look at the intersections of the arcs $\Gamma_* \mid U_i$ and $\Gamma_* \mid U_j$, for $i \neq j$. Let $u_0 \in U_i$ and $u_1 \in U_j$ be given with $P = \Gamma_*(u_0) = \Gamma_*(u_1)$.

If P is a transversal self intersection of Γ_*, then $\gamma(t)$ must have had a transversal self intersection near P, just before t_{Max}. Since the number of self intersections of $\gamma(t)$ is finite (and nonincreasing with time), there can only be a finite number of such (u_0, u_1)'s.

In general we have the following alternative.

Lemma 6.2. *If $u_0 \in U_i, u_1 \in U_j$ $(i \neq j)$ and $P = \Gamma_*(u_0) = \Gamma_*(u_1)$, then P is either an isolatated self intersection of γ^*, or there is an $\epsilon > 0$ such that*

$$\Gamma_*([u_0, u_0+\epsilon)) \subset \Gamma_*(U_j) \text{ or } \Gamma_*((u_0-\epsilon, u_0]) \subset \Gamma_*(U_j)$$

holds.

Proof. Indeed, if the second alternative doesn't occur, then there is an interval $(a,b) \subset U_i$ containing u_0 for which $\Gamma_*(a)$ and $\Gamma_*(b)$ do not lie on $\Gamma_*(U_j)$.

Since P is not one of the Q_i's, there are at most a finite number of v's in U_j with $\Gamma\ (v) = P$; let v_0 be one of these. Choose an interval $(c, d) \subset U_j$ containing v_0, such that $\Gamma_*(c)$ and $\Gamma_*(d)$ do not lie on $\Gamma_*(U_i)$. By continuity of Γ there is a $t_0 < t_{Max}$ such that $\Gamma(a,t)$ and $\Gamma(b,t)$ do not lie on $\Gamma(U_j,t)$ for $t_0 < t < t_{Max}$, and vice versa. It follows that for all $t \in [t_0, t_{Max}]$ the number of intersections of the two arcs $\Gamma(U_i,t)$ and $\Gamma(U_j,t)$ will be finite and nonincreasing in t. Therefore P must be an isolated intersection of $\Gamma_*(U_i)$ and $\Gamma_*(U_j)$. $\square$

Consider a nonisolated self intersection $P = \Gamma_*(u_0) \in \Gamma_*(U_j)$, and let $E \subset U_i$ be the largest interval (with or without end points) containing u_0, such that $\Gamma_*(E) \subset \Gamma_*(U_j)$.

Lemma 6.3. *Either $E = U_i$, or, for some $c \in U_i$, one has $E = (a_i, c\,]$ or $E = \lfloor c, b_i)$.*

(where $U_i=(a_i, b_i)$).

Proof. First note that E is closed in U_i. Indeed, if $\{u_n \in E\}$ converges to some $u_0 \in U_i$, then there are $v_n \in U_j$ with $\Gamma_*(u_n)=\Gamma_*(v_n)$. By passing to a subsequence we can arrange that the v_n's also converge to some $v_0 \in \bar{U}_j$. Since $\Gamma_*(v_0)=\Gamma_*(u_0)$ is not one of the Q_i's, v_0 must lie in U_j (i.e., not on ∂U), so that u_0 belongs to E.

Since E is a closed interval in U_i, the lemma states that E is not of the form $[c, d] \subset U_i$. Suppose that E does have this form. Then, by the maximality of E, there is a $\delta>0$ such that $E_\delta=[c-\delta, d+\delta] \subset U_i$, and $\Gamma_*(\partial E_\delta)$ is disjoint from $\Gamma_*(\bar{U}_j)$. By continuity of Γ, this also holds on some short time interval $t_0 \leq t \leq t_{Max}$, so that the variation on theorem 1.3 implies that $\Gamma_*(E_\delta)$ and $\Gamma_*(U_j)$ only have a finite number of intersections, a clear contradiction with $\Gamma_*(E) \subset \Gamma_*(U_j)$. □

Remark. It is not clear whether or not a nonisolated self intersection can occur. If such a self intersection developes, then the limit curve contains an arc which is traversed at least twice by the limit parametrisation Γ_*. It follows from the strong maximum principle that any neighborhood of any point on this arc contains a self intersection of one of the $\gamma(t)$'s, for some $t<t_{Max}$. In other words, one or more self intersections of $\gamma(t)$ have to move up and down the doubly parametrized part of the limit curve *infinitly often* before the curve becomes singular. So, to say the least, a nonisolated self intersection is a slightly pathological thing. Nevertheless, we can't conclude from the arguments in this section that they don't occur.

On the other hand there is no known example of a solution whose limit curve contains a doubly parametrized arc. An indication that they might occur is given by the existence of solutions $u(t, x)$ of the ordinary heat equation

$$u_t = u_{xx}$$

on the rectangle $\{0<x<1, 0<t<\infty\}$ which vanish in finite time (see Matano's paper [Ma].)

Assuming analyticity of V, the manifold M and the metric g improves the situation a little bit. In this case the curves $\gamma(t)$ are known to be real analytic, and even the smooth parts of the limit curve are real analytic. Indeed, one could follow a compact sub arc of $\gamma(t)$ whose limit doesn't contain any of the singular points Q_j, and simply continue it beyond t_{Max}, by requiring it to have normal velocity $v^{\perp}=V(t, k)$, while keeping the end points fixed. Then for each t this sub arc will be an analytic arc, although, as is usually the case with solutions of parabolic equations, it will not necessarily be analytic in the time variable. But if the limit curve is piecewise real analytic then the second alternative in lemma 6.3 cannot occur!

Nevertheless, even if the situation in the real analytic case is somewhat simpler, one still can't exclude the possiblity of a "figure eight" collapsing to a smooth arc (see figure 7.1.)

We shall now define the reduced limit curve by removing certain "redundant parts" from Γ_*. First we'll collapse the intervals I_j. Let S' be the circle S^1 with the intervals I_j identified to points; i.e., call two points $x, y \in S^1$ equivalent if $x=y$, or if they lie in the same interval I_j, and let S' be the quotient of S^1 with respect to this equivalence relation. Since Γ_* is constant on each of the intervals I_j it defines a map from S' to M which we shall also denote by Γ_* (so $\Gamma_*([x])=\Gamma_*(x)$ for any $x \in S^1$ and its corresponding equivalence class $[x] \in S'$).

In general one should not expect $\Gamma_*: S' \to M$ to be locally injective. However, if Γ_* fails to be locally injective then lemma 6.3 tells us that for at least one of the intervals $I_j=[a_j, b_j]$ there must be $\epsilon_1, \epsilon_2>0$ and a homeomorphism

$$\eta:[a_j-\epsilon_1, a_j] \to [b_j, b_j+\epsilon_2]$$

for which $\Gamma_*(\eta(u))=\Gamma_*(u)$. In other words, if $Q_j=\Gamma_*([I_j])$ then as, u varies from $a_j-\epsilon_1$ to $b_j+\epsilon_2$, the point $\Gamma_*(u)$ makes its way from $\Gamma_*(a_j-\epsilon_1)$ to Q_j and then retraces its path back to $\Gamma_*(b_j+\epsilon_2)=\Gamma_*(a_j-\epsilon_1)$.

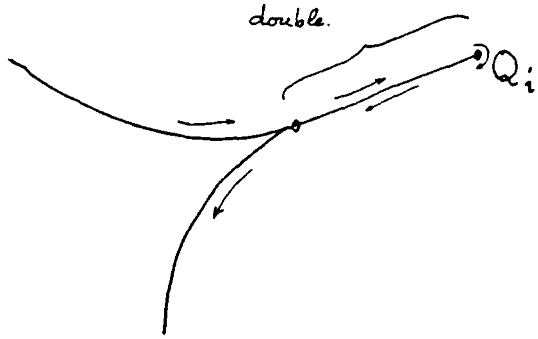

Figure 6.2— A redundant part of Γ_*.

For any interval $I_j = [a, b]$ there will be a unique largest interval $[c, d] = E_j \supset I_j$ for which such a homeomorphism $\eta : [c, a] \to [b, d]$ exists. Choose a maximal disjoint subcollection $E_1, \cdots, E_m$ of the E's and define S'' to be the circle S^1 with the intervals E_i $(1 \le i \le m)$ collapsed to points $[E_1], \cdots, [E_m]$. Since Γ_* is constant on each ∂E_j we can define a reduced limit curve $\Gamma_{red} : S'' \to M$ by

$$\Gamma_{red}(u) = \Gamma_*(u) \text{ if } u \notin E_j,\ j=1, \cdots, m$$

$$= \Gamma_*(v_j) \text{ if } u \in E_j \text{ and } v_j \in \partial E_j.$$

The collection of intervals E_j was chosen maximal so that Γ_{red} is a local homeomorphism of S'' into M. Away from the points $[E_j] \in S''$ Γ_{red} is smooth $(C^{2,\alpha})$, and its total absolute curvature is finite. Thus Γ_{red} is a piecewise C^1 curve. Moreover all self intersections of the reduced limit curve are isolated (since the non isolated ones are contained in the E_j's), and because such isolated intersections existed just before the curves $\gamma(t)$ became singular, Γ_{red} cannot have more self intersections than any of the $\gamma(t)$'s had $(0 < t < t_{Max})$.

Lemma 6.4. *Any piecewise C^1 map from S^1 to M which is locally injective parametrizes a C^1 locally graph like curve.*

In particular, our reduced limit curve Γ_{red} is C^1 locally graphlike and we can use it as initial data for the equation $v^{\perp} = V(t, k)$.

Proof. Let $u \in S''$ be given. If u is not one of the $[E_j]$ then Γ_{red} is smooth with nonvanishing derivative near u, so that the reduced curve certainly is a C^1 regular curve around u.

If $u = [E_j]$, then for some $\epsilon > 0$ Γ_{red} will be C^1 on $\Big[[E_j] - \epsilon, [E_j]\Big]$ and $\Big[[E_j], [E_j] + \epsilon\Big]$. In particular the left and right hand limits of the unit tangent to Γ_{red}, t_- and t_+, will exist.

If $t_- \ne -t_+$, then near $\Gamma_{red}([E_j])$ the reduced curve will be locally Lipschitz, and therefore certainly C^1 locally graphlike.

In the remaining case, $t_- = -t_+$, in which Γ_{red} makes a 180 degree turn at $[E_j]$, one can choose smooth coordinates x, y near the point $\Gamma_{red}([E_j])$ such that near this point the image of Γ_{red} is the union of the graphs of two C^1 functions $y = f_1(x)$ and $y = f_2(x)$, both defined for $0 \le x \le 1$. Since Γ_{red} has at most a finite number of self intersections we may assume that the two graphs are disjoint, except at $\Gamma_{red}([E_j])$. Thus f_1 and f_2 satisfy

$$f_1(x) < f_2(x) \text{ for } 0 < x \le 1$$

$$f_1(0) = f_2(0) = f_1'(0) = f_2'(0) = 0.$$

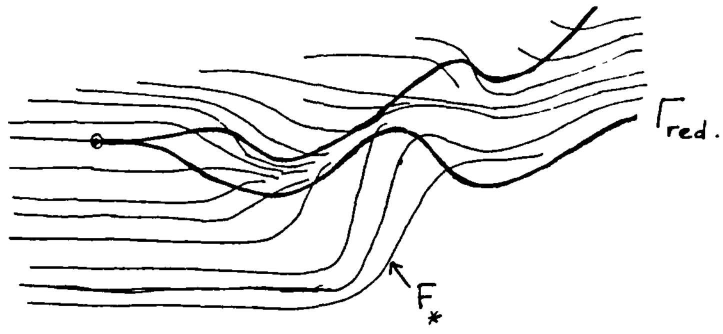

Figure 6.3— A 180 degree turn.

Now choose a continuous function $\eta:[-1,1]\rightarrow\mathbf{R}$ which vanishes for $x\le 0$, and which satisfies $\eta(x) > \max(\,|f_1'(x)|\,,\,|f_2'(x)|\,)$ for $0<x\le 1$. Define

$$a(x,y) = \frac{f_1(x)+f_2(x)-2y}{f_2(x)-f_1(x)}\,\eta(x) \text{ if } f_1(x)<y<f_2(x).$$

$$= +\eta(x) \text{ if } y\ge f_2(x) \text{ and } x>0,$$

$$= -\eta(x) \text{ if } y\le f_1(x) \text{ and } x>0,$$

$$= 0 \quad \text{if } x\le 0.$$

Choose some $x_0>0$, and let y_p be the solution of the ODE

$$(6.1)\qquad \frac{dy}{dx} = a(x,y(x)),\quad y_p(-x_0)=p.$$

We have chosen the function η such that

$$a(x,f_2(x)) < f_2'(x)$$

$$a(x,f_1(x)) > f_1'(x)$$

holds for $0<x\le 1$, and this implies that the graphs of the y_p's intersect the curve segment $\Gamma_{red}(\left[[E_j]-\epsilon,[E_j]+\epsilon\right])$ at most once.

Using $a_y(x,y)\le 0$, one easily verifies that $y_p(x)$ is uniquely determined by its initial value p, and that $\sigma(q,p)=(q,y_p(q))$ is a C^1 map. Since the y_p's are constant for $x\le 0$, there is exactlyone p such that the graph of y_p passes through the origin (namely, $p=0$). Away from the origin a_y is locally bounded, so the existence and uniqueness theorem for ODE's implies that the graphs of the y_p's are disjoint; hence σ is also a local homeomorphism. Since its derivative $\sigma_q=(1,a(q,p))$ never vanishes, the reduced limit curve is C^1 locally graphlike. □

7. Generalized solutions.

We still assume that $V:S^1(M)\times\mathbf{R}\rightarrow\mathbf{R}$ is $C^{2,1}$, and that it satisfies the conditions $V_1,\,\cdots,V_5^*$ as well as the symmetry condition S.

Let $\gamma_0 \in \Omega(M)$ be a regular curve, and let $\gamma(t)$ be the corresponding maximal solution of $v^\perp = V(t,k)$. If this solution becomes singular in finite time, say at $t = t_1$, then we have just shown that there exists a reduced limit curve $\gamma(t_1)$, which is C^1 locally graphlike. This curve may be empty, e.g. if the solution $\gamma(t)$ shrinks to a point, or if $\gamma(t)$ is a figure eight which collapses to a curve segment[3] (see figure 7.1). If this curve is not empty however, then we can use it as an initial value for the parabolic equation. This gives us a new maximal classical solution $\gamma(t)$ $(t_1 \le t_2)$. The new solution may exist forever $(t_2 = \infty)$, or it too may become singular in finite time $(t_2 < \infty)$. Again, if the new reduced limit curve is not empty, then we can continue the solution after t_2.

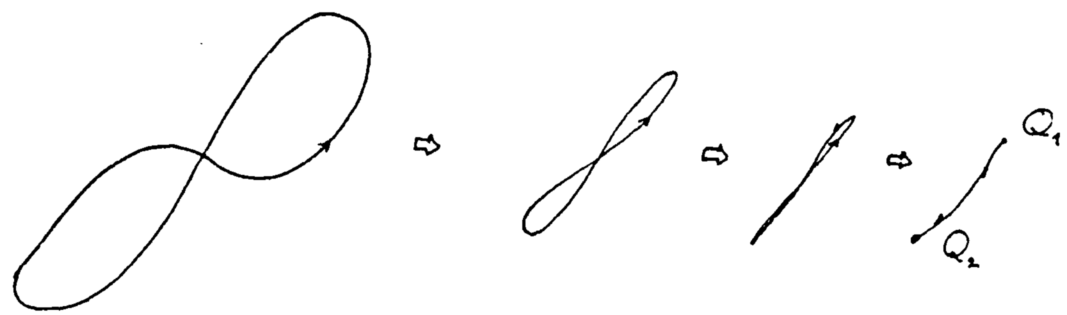

Figure 7.1 — A collapsing "figure eight," with an empty reduced limit curve.

This procedure can be repeated indefinitely, and as a result one obtains either a finite sequence of maximal classical solutions of which the last has an infinite lifespan, or an empty reduced limit curve, or else one obtains an infinite sequence of solutions $\gamma : (t_i, t_{i+1}) \to \Omega(M)$. We shall call such a sequence (finite or infinite) of classical solutions a *generalized solution* of $v^\perp = V(t,k)$.

With what we have proven so far we cannot exclude that the sequence t_n at which the maximal solutions become singular might be bounded, i.e. that $\sup_{n \ge 1} t_n = t_\infty < \infty$. The main result of this section says that this cannot happen.

Theorem 7.1. *Let $\gamma(t)$ $(0 \le t \le t_\infty)$ be a generalized solution of $v^\perp = V(t,k)$. At each time t_n at which $\gamma(t)$ becomes singular either*

$$\liminf_{t \to t_{Max}} \alpha_\epsilon(\gamma(t)) \ge \pi$$

and the limit curve γ^* has less self intersections than $\gamma(t)$, or else

$$\liminf_{t \to t_{Max}} \alpha_\epsilon(\gamma(t)) \ge 2\pi.$$

In other words the only way a classical solution can become singular is by forming a kink of at least 180 degrees in which a self intersection disappears, or else by forming a kink of at least 360 degrees.

Corollary. *For any $\gamma_0 \in \Omega(M)$ there is a generalized solution which exists for all time, or else there is a generalized solution which has an empty reduced limit curve.*

3. As in section six, I don't know whether this can happen or not. The results of this paper don't exclude the possibility of a "collapsing figure eight," but on the other hand I don't know of examples of initial curves that produce such a collapsing figure eight.

It follows from Grayson's work on the curve shortening equation [Gr2] that if $V(t,k) \equiv k$, then the only way in which a curve can become singular is by contracting a loop, thereby losing a self intersection. It seems reasonable to conjecture that this is also true in the general case at which we are looking, and that one could prove this if one had a generalization of his "δ-whisker lemma." In fact, in his paper [Gr1, p. 300] Grayson calls this "Surprisingly, . . . the easiest case to rule out."

Proof of theorem 7.1. We choose a normal parametrisation $\Gamma : (0, t_{Max}) \times S^1 \to M$ of the given classical solution $\gamma : (0, t_{Max}) \to \Omega(M)$, i.e. a parametrisation for which $\Gamma_t(t,u) \perp \Gamma_u(t,u)$ holds for all (t,u). Just as in section 5 we let $\mathbf{K}_t$ be the Borel measure on M given by the total absolute curvature of $\gamma(t)$.

Let $Q \in M$ be one of the singular points of the limit curve γ^*. Assume that for some ϵ_0 and $0 < t_1 < t_{Max}$ the curve $\gamma(t)$ has no self intersection in $\mathbf{N}_{\epsilon_0}(Q)$, for any $t \in (t_1, t_{Max})$. Then we'll show by contradiction that for any $\epsilon > 0$ one has

$$\liminf_{t \to t_{Max}} \alpha_\epsilon(\gamma(t) \cap \mathbf{N}_{\epsilon_0}(Q)) \geq 2\pi.$$

Suppose that this is not true. Then there's a sequence $t_n \to t_{Max}$ and an $\epsilon_1 > 0$ such that

$$\alpha_{\epsilon_1}(\gamma(t_n) \cap \mathbf{N}_{\epsilon_0}(Q)) \leq 2\pi - \alpha$$

for some small $\alpha > 0$ — we'll assume that $\alpha < 10^{-3} \times \pi$.

Since $\gamma(t)$ is simple in $\mathbf{N}_{\epsilon_0}(Q)$, and since $\gamma(t)$ converges in the $C^{2,\alpha}$ topology, away from the point Q, it follows from the strong maximum principle that $\gamma^* \cap \mathbf{N}_\epsilon(Q)$ consists of a finite number of C^1 curve segments whose only common point is Q. Away from this point these segments are even $C^{2,\alpha}$ smooth.

Without loss of generality we may assume that $\gamma(t_n) \cap \mathbf{N}_{\epsilon_0}(Q)$ has only one component; if there are more, then one should apply the following arguments to each of these components.

We'll also assume that ϵ_1 is so small that $\mathbf{N}_{\epsilon_1}(Q)$ fits in a coordinate neighbourhood of Q. Let (x,y) be isothermal coordinates on $\mathbf{N}_{\epsilon_1}(Q)$, so that the metric of M has the form

$$(ds)^2 = \rho(x,y)[(dx)^2 + (dy)^2]$$

for some smooth function $\rho(x,y) > 0$. By choosing ϵ_1 small enough we can ensure that ρ is nearly constant; we shall in fact assume that $1 \leq \rho(x,y) \leq 1 + 10^{-14}$.

For $p \in \mathbf{N}_{\epsilon_1}(Q)$ we let $e_x(p) = \rho(p)^{-1/2}\, \partial_x$ and $e_y(p) = \rho(p)^{-1/2}\, \partial_y$ be the unit vector fields in the x and y directions. Define

$$A = \sup_{\mathbf{N}_{\epsilon_0}(Q)} (\, |\nabla e_x| + |\nabla e_y| \,).$$

For any $p \in \gamma(t_n) \cap \mathbf{N}_{\epsilon_1}(Q)$ define $\theta_n(p)$ to be the angle between $t_{\gamma(t_n)}(p)$ and $e_x(p)$. It follows from $\cos(\theta_n) = (e_x, t_{\gamma(t_n)})$, and $\sin(\theta_n) = (e_y, t_{\gamma(t_n)})$, that

$$\frac{d\theta_n}{ds} = k_{\gamma(t_n)} + (t, \cos(\theta_n)\nabla_t e_y - \sin(\theta_n)\nabla_t e_x)$$

and thus

$$(7.1) \qquad \left| \frac{d\theta_n}{ds} - k_{\gamma(t_n)} \right| \leq A.$$

Now we assume that ϵ is so small that

(7.2^{a}) $\quad \epsilon<\epsilon_1$

(7.2^{b}) $\quad \mathbf{K}_{t_{Max}}(\mathbf{N}_\epsilon(Q)-\{Q\}) < 10^{-37}\times\alpha$

(7.2^{c}) $\quad length\,(\gamma^*\cap\mathbf{N}_\epsilon(Q)) < 10^{-37}\times\alpha/A$

holds.

Since the length of $\gamma(t_n)\cap\mathbf{N}_\epsilon(Q)$ converges to the length of $\gamma^*\cap\mathbf{N}_\epsilon(Q)$, the last condition on ϵ implies that for sufficiently large n one has

(7.3) $\quad length\,(\gamma(t_n)\cap\mathbf{N}_\epsilon(Q)) < \frac{\alpha}{2A}.$

Then it follows from (7.1) that for any two points $p, p'\in\gamma(t_n)\cap\mathbf{N}_\epsilon(Q)$

$$|\theta_n(p)-\theta_n(p')| \le |\int_{p'}^{p} k(s)\,ds| + A\times length\,(\gamma(t_n)\cap\mathbf{N}_\epsilon(Q)) \le 2\pi-\frac{\alpha}{2}.$$

which gives an estimate for the oscillation of θ_n on $\gamma(t_n)\cap\mathbf{N}_\epsilon(Q)$:

(7.4) $$\operatorname*{osc}_p \theta_n(p) \overset{def}{=} \sup_p \theta_n(p) - \inf_p \theta_n(p) \le 2\pi-\frac{\alpha}{2}.$$

By passing to a subsequence, and rotating the coordinates x and y, if necessary, one can ensure that the angle θ_n satisfies

$$-\frac{\pi}{2}+\frac{\alpha}{8} \le \theta_n(p) \le \frac{3\pi}{2}-\frac{\alpha}{8}$$

for any $p\in\gamma(t_n)\cap\mathbf{N}_\epsilon(Q)$. Since $\gamma(t_n)$ converges in C^1 to γ^*, away from Q, this also implies

$$-\frac{\pi}{2}+\frac{\alpha}{8} \le \theta_*(p) \le \frac{3\pi}{2}-\frac{\alpha}{8}$$

for all $p\in\gamma^*\cap\mathbf{N}_\epsilon(Q), p\neq Q$, where $\theta_*(p)$ is the angle between $t_{\gamma^*}(p)$ and $e_x(p)$.

Put $\beta=10^{-3}\times\alpha$. Then for each $p\in\gamma(t_n)\cap\mathbf{N}_\epsilon(Q)$ we have

$$\max(-\frac{\pi}{2}+2\beta, \theta_n(p)-\pi+\beta) < \min(\frac{\pi}{2}-2\beta, \theta_n(p)-\beta),$$

since $\beta=10^{-3}\alpha < 10^{-6}\pi$. Therefore there is a continuous function ϕ_n defined on the segment $\gamma(t_n)\cap\mathbf{N}_\epsilon(Q)$, which satisfies

(7.5) $\quad \theta_n(p)-\pi+\beta \le \phi_n(p) \le \theta_n(p)-\beta$

(7.6) $\quad |\phi_n(p)| \le \frac{\pi}{2}-2\beta,$

and, by Tietze's theorem, we can extend this function to the entire disk $\mathbf{N}_\epsilon(Q)$, so that it still satisfies (7.6). After slightly perturbing ϕ_n we may assume that it is a C^∞ smooth function. Since $\gamma(t)$ converges in $C^{2,\alpha}$ to γ^*, at least away from the point Q, we can choose the ϕ_n so that all their derivatives of any order are uniformly bounded on any compact $K \subset \mathbf{N}_\epsilon(Q)$ which does not contain the point Q. By passing to a subsequence we may even assume that the ϕ_n and their derivatives converge uniformly on such compact sets. We'll denote the limit by ϕ_*.

The foliations $\mathbf{F}_n$ and $\mathbf{F}_*$. Consider the vectorfield

$$X_n(p) = \cos\phi_n(p)e_x(p) + \sin\phi_n(p)e_y.$$

The integral curves of this vectorfield form a foliation $\mathbf{F}_n$ of the disk $\mathbf{N}_\epsilon(Q)$. The functions ϕ_n and ϕ_* were defined in such a way that the following lemma is true.

Lemma 7.2. *The leaves of $\mathbf{F}_n$ intersect $\gamma(t_n)$ transversally; in fact, whenever a leaf of the foliation and the curve $\gamma(t_n)$ intersect, the angle of intersection is at least β. At any point p on $\gamma(t_n)$ the pair $\{X_n(p), t_{\gamma(t_n)}(p)\}$ is a positively oriented basis for $T_p(M)$.*

Here we've assumed that our coordinates are chosen so that $\{e_x, e_y\}$ is positively oriented. The lemma follows immediately from the inequalities (7.5).

As $n \to \infty$ the vectorfields X_n converge to $X_* = \cos(\phi_*)e_x + \sin(\phi_*)e_y$, on $\mathbf{N}_\epsilon(Q) - \{Q\}$. The limiting vector field defines a smooth foliation $\mathbf{F}_*$ of the punctured disk $\mathbf{N}_\epsilon(Q) - \{Q\}$.

Construction of the box R. Let $l \subset \mathbf{N}_\epsilon(Q)$ be the vertical linesegment with end points $P_\pm = (0, \pm\epsilon/2)$, and let $\lambda_\pm$ be the leaf of the limit foliation $\mathbf{F}_*$ through the point $P_\pm$. Both λ_+ and λ_- are graphs of Lipschitz continuous functions $y = f_\pm(x)$, and since $|f_\pm{}'(x)| \le \cot 2\beta$, there's a positive δ such that $f_\pm$ are both defined on the interval $-\delta \le x \le \delta$. In fact, since the metric on $\mathbf{N}_{\epsilon_1}(Q)$ is almost flat $(1 \le \rho \le 1 + 10^{-14})$, we can take $\delta = 0.5 \times 10^{-6} \times \alpha\epsilon$ (see figure 7.2.) From here on we shall use $\lambda_\pm$ to denote the part of these leaves which lies in the strip $|x| \le \delta$.

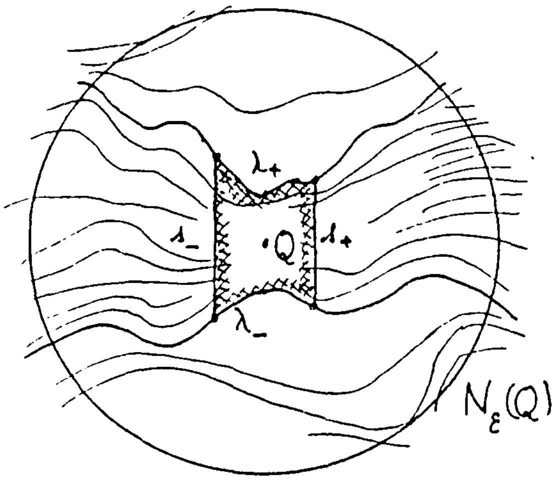

Figure 7.2.— The nonlinear rectangle **R**.

Define

$$\mathbf{R} = \{(x, y) \mid -\delta \le x \le \delta, f_-(x) \le y \le f_+(x)\}$$

The region **R** is a nonlinear rectangle; its left and right hand sides are the vertical line segments:

$$s_+ = \{\delta\} \times [f_-(\delta), f_+(\delta)]; \qquad s_- = \{-\delta\} \times [f_-(-\delta), f_+(-\delta)].$$

Claim. *We can assume that γ^* is disjoint from the two sides $s_\pm$ of the rectangle* **R**.

Proof of the claim. Suppose not. By assumption the complement $\gamma^* - \{Q\}$ of Q in γ^* consists of two components, γ_1^* and γ_2^*. Assume that the curve γ^* is oriented so that γ_1^* is directed towards Q,

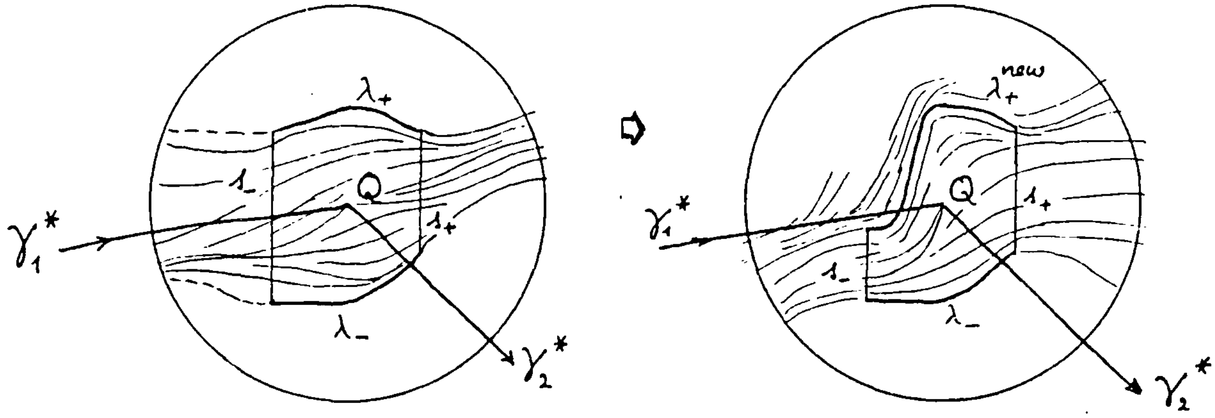

Figure 7.3 — How to change λ_+, and F_*

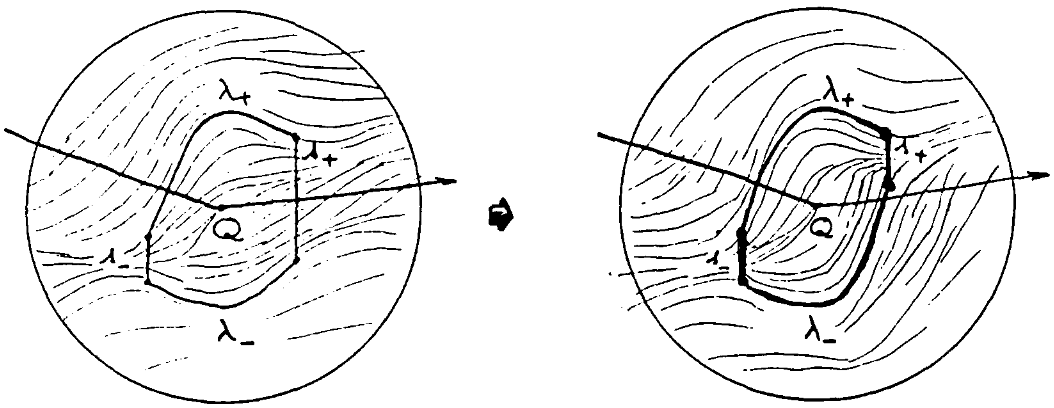

Figure 7.4 — How to change λ_- and F_*.

and γ_2^* is directed away from Q (outside of its singular points γ^* gets its orientation from being the C^1 limit of the oriented curves $\gamma(t)$.) If γ_j^* intersects one of the two sides $s_\pm$, then there must be a point $p_j \in \gamma_j^*$ such that

$$|\tan(\theta_*(p_j))| \leq \frac{\epsilon}{\delta} \leq 2\times 10^6 \times \alpha^{-1},$$

and hence

$$|\theta_*(p_j)| \leq \frac{\pi}{2} - 0.5\times 10^{-6}\times\alpha.$$

Our assumptions (7.2^b) and (7.2^c) imply that the oscillation of θ_* on each of the $\gamma_j^* \cap \mathbf{N}_\epsilon(Q)$'s is less than $2\times 10^{-37}\times\alpha$. Therefore we find that

$$|\theta_*(p)| \leq \frac{\pi}{2} - 10^{-7}\times\alpha$$

for any $p \in \gamma_j^* \cap \mathbf{N}_\epsilon(Q)$.

So we see that, if γ_j^* intersects one of the sides of $\mathbf{R}$, then $\gamma_j^* \cap \mathbf{N}_\epsilon(Q)$ is the graph of a function whose derivative can be estimated by $\cot(10^7/\alpha)$.

We must distinguish between two cases: (1) each side $s_\pm$ intersects at most one of the components γ_j^*, or, (2) both components go through one of the two sides $s_\pm$.

First case. Let γ_1^* intersect the left hand side s_-. Then one can change ϕ_* in the region $-\delta \leq x \leq \delta/2$, $y < f_+(x)$, so that the new λ_- bends upwards and intersects γ_1^* (see figure 7.3.) One can choose the new ϕ_* so that it too satisfies

$$\sup_{\mathbf{N}_\epsilon(Q)} |\phi_*(p)| < \frac{\pi}{2}.$$

and so that the leaves of the new $\mathbf{F}_*$ are again graphs of Lipschitz functions.

Similarly, if γ_2^* intersects the righthand side s_+, then one can modify ϕ_* in $\delta/2 \leq x \leq \delta$, $f_-(x) < y$, so that the new λ_+ bends down, and intersects γ_2^* (see figure 7.4.)

If both sub cases occur at the same time, then one makes both changes. The other ways in which the "first case" can occur are: γ_1^* passes through s_+, γ_2^* passes through s_-, or both. In each of these three cases one can make similiar changes.

Second case. Suppose that both γ_j^*'s intersect the left hand side s_-. Then γ_2^* must lie above γ_1^* (see figure 7.5.) Indeed, γ^* is the limit of the *simple* C^1 curves $\gamma(t_n)$ whose tangents never point downwards, i.e. $t_{\gamma(t_n)}(p) \neq -e_y$ for any $p \in \gamma(t_n) \cap \mathbf{N}_\epsilon(Q)$, which follows from (7.3). If γ_2^* would lie below γ_1^*, then any curve *without self intersections* which is C^1 close to γ^*, away from Q, must have a point at which the tangent is $(0, -1)$. So this can't happen.

Given that γ_2^* lies above γ_1^*, it is again clear that one can change ϕ_* in the region $-\delta \leq x \leq -\delta/2$ so that λ_+ gets bent down, and λ_- gets bent up in such a way that γ_2^* (γ_1^*) intersects the new λ_+ (λ_-). See figure 7.6.

Of course, if the γ_j^* go through the other side, then one can again do the same kind of construction.

So, we may indeed assume that γ^* does not intersect the sides of the rectangle $\mathbf{R}$. □

In the proof of the claim we have changed ϕ_*, so that the ϕ_n will not converge to ϕ_* anymore. However, we have only changed ϕ_* outside of the open set $U = int(\mathbf{N}_{\delta/2}(Q))$, so we can change the ϕ_n's so that they will again converge to ϕ_*. Let $\eta \in C_c^\infty(U)$ be a function for which $\eta(p) \equiv 1$ in a

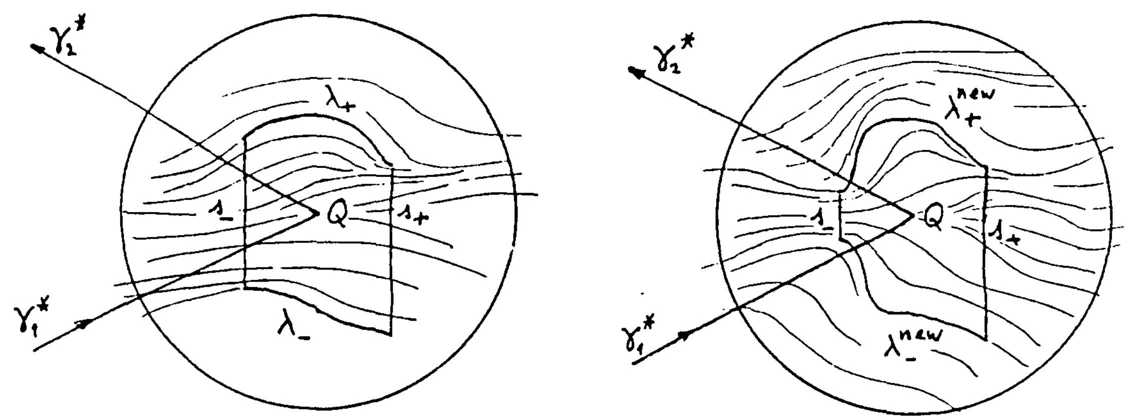

Figure 7.5 When both γ_j^* go through s_-.

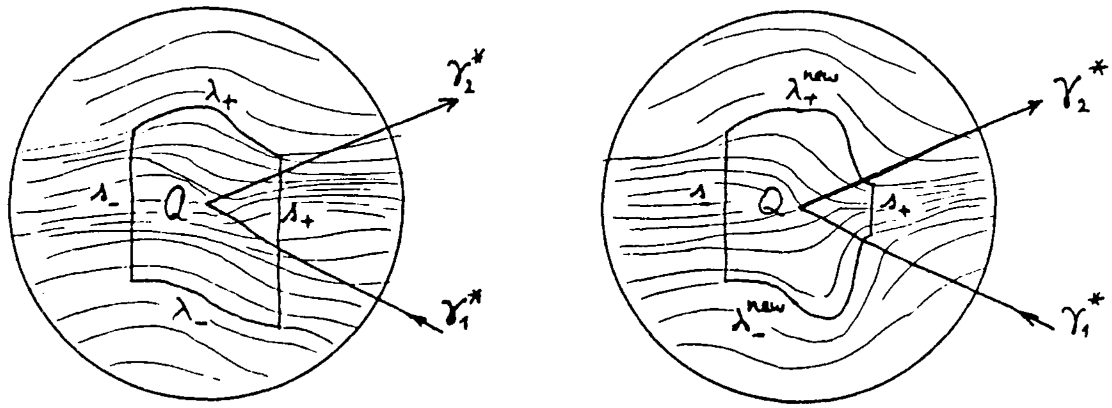

Figure 7.6 When both γ_j^* go through s_+.

neighbourhood of Q, and define

$$\phi_n^{new}(p) = \eta(p)\phi_n(p)+(1-\eta(p))\phi_*^{new}(p),.$$

Then ϕ_n and ϕ_n^{new} coincide in a neighbourhood of Q, and ϕ_n will converge to the new ϕ_*. From here on we'll write ϕ_n for ϕ_n^{new}.

Conclusion of the proof of theorem 7.1. Let $\lambda_{n,\xi}$ be the leaf of F_n which goes through the point $(0, \xi)$. We want to let $\lambda_{n,\xi}$ evolve with normal velocity $v^\perp = V(t,k)$, but just as in section four, we don't have an existence theorem. Therefore we must modify V again.

Since $\gamma(t) \rightarrow \gamma^*$ in the Hausdorff metric, one can find an open neighborhood $O \supset s_+ \cup s_-$, and an $n_0 \geq 1$ such that $\gamma(t)$ is disjoint from O for all $t \geq t_{n_0}$. Choose a smaller neighborhood $O' \subset O$ of $s_+ \cup s_-$, and change V in $S^1(O) \times \mathbf{R}$ so that $V^{new}(t,k) \equiv k$ for all k and $t \in S^1(O')$. Then $\gamma(t)$ also evolves with normal velocity $v^\perp = V^{new}(t_{\gamma(t)}, k_{\gamma(t)})$, for $t_{n_0} \leq t < t_{Max}$.

Let $\lambda_{n,\xi}(t)$ $(t \geq t_n)$ evolve with normal velocity $v^\perp = V^{new}(t_{\lambda_{n,\xi}(t)}, k_{\lambda_{n,\xi}(t)})$, with initial configuration $\lambda_{n,\xi}(t_n) = \lambda_{n,\xi}$, and with *fixed* end points. Since the $\lambda_{n,\xi}$ are uniformly Lipschitz, there is a $\tau > 0$ for which the $\lambda_{n,\xi}(t)$ exist for $t_n \leq t \leq t_n + \tau$. The $\lambda_{n,\xi}(t)$ will form a foliated rectangle $\mathbf{R}^n(t)$.

Choose an $n \geq n_0$ with $t_n + \tau > t_{Max}$. If n is large enough, then it follows from the displacement estimates of section five in part I that the rectangle $\mathbf{R}^n(t_{Max})$ will contain the point Q.

The curve $\gamma(t_n)$ intersects each leaf $\lambda_{n,\xi}$ exactly once and transversally. The end points of the $\lambda_{n,\xi}(t)$ lie on the sides $s_\pm$, and therefore the curve $\gamma(t)$ stays away from these end points for $t_n \leq t < t_{Max}$. This implies that $\gamma(t)$ will be transversal to the $\lambda_{n,\xi}(t)$ for all $t \in (t_n, t_{Max})$.

Now we can use the same trick as in section 4. We introduce a second foliation whose leaves $\mu_{n,\xi}(t)$ evolve with $v^\perp = V(t,k)$, and whose end points are also fixed on the sides $s_\pm$. The initial position $\mu_{n,\xi}$ of $\mu_{n,\xi}(t)$ is obtained from $\lambda_{n,\xi}$ by "bending the ends" of $\lambda_{n,\xi}$ as in figure 7.7.

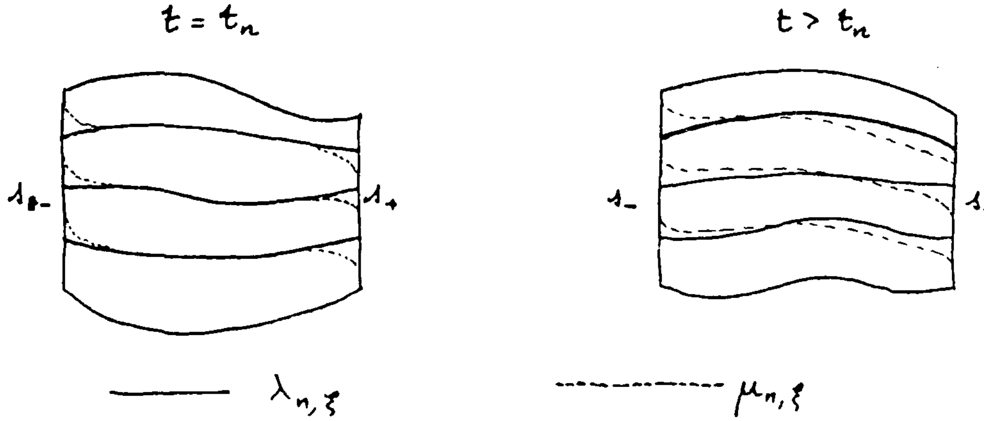

Figure 7.7.— The two foliations with leaves $\lambda_{n,\xi}$ and $\mu_{n,\xi}$,
at $t = t_n$ and a little later.

If the $\mu_{n,\xi}$ are chosen close enough to $\lambda_{n,\xi}$ (in the C^∞ topology) then the $\mu_{n,\xi}(t)$ will also exist as $C^{2,\alpha}$ curves for $t_n \leq t \leq t_{Max}$. The result is that the $\lambda_{n,\xi}(t)$ will be transversal to the $\mu_{n,\xi}(t)$, while the $\gamma(t)$ will be transversal to both the $\lambda_{n,\xi}(t)$ and $\mu_{n,\xi}(t)$ foliations. Since the angle of intersectection between $\lambda_{n,\xi}(t)$ and $\mu_{n,\xi}(t)$ is bounded away from zero, for $t \geq (t_n + t_{Max})/2$, this means that the $\gamma(t)$'s remain uniformly locally Lipschitz as t tends to t_{Max}. But this contradicts our initial assumption that $\gamma(t)$ becomes singular at Q. $\square$

Proof of the corollary. We know that the total absolute curvature of any solution cannot grow faster than exponentially, and that the number of self intersections of a solution immediately becomes finite and does not increase.

We claim that theorem 7.1 and its proof show that the following is true. Whenever a solution becomes singular and the limit curve has the same number of self intersections as some $\gamma(t)$, then the total absolute curvature K_* of the limit curve satisfies

$$(7.7) \qquad K_* \leq \lim_{t \to t_{Max}} K(t) - \pi,$$

where $K(t)$ is the total absolute curvature of $\gamma(t)$.

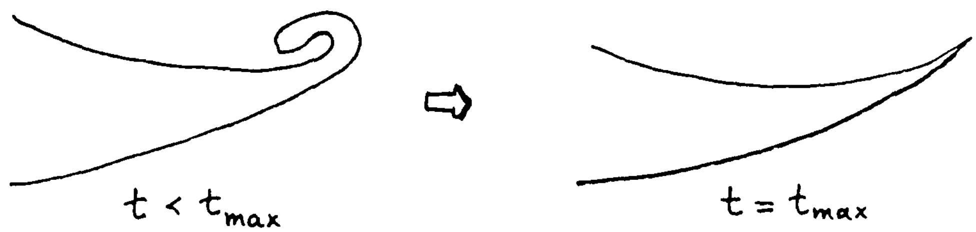

Figure 7.8 — How the total curvature drops by π.

Indeed, let Q be a singular point of the limit curve. As in the proof of the theorem we may assume that, if ϵ is small enough, $\gamma(t) \cap \mathbf{N}_\epsilon(Q)$ has only one component. Given any $\delta > 0$ there's an $\epsilon > 0$ such that the total absolute curvature of $\mathbf{N}_\epsilon(Q) \cap \gamma^*$ is less than $\pi + \delta$. By the theorem, for t close to t_{Max}, the total absolute curvature of $\mathbf{N}_\epsilon(Q)$ must be at least $2\pi - \delta$.

So, assuming that $\gamma(t)$ doesn't lose any self intersections at t_{Max}, we find that at least $\pi - 2\delta$ of curvature disappears into one of the singular points as $t \to t_{Max}$. Since $\gamma(t)$ converges in C^2 away from the singular points, this proves (7.7).

If a generalized solution would have an infinite number of singularities in a finite time interval, then it would lose an infinite amount of total absolute curvature, or it would lose an infinite number of self intersections. Of course neither of these two possibilities can happen, so that the generalized solution must exist forever. □

8. Curves without nodes.

Let $\gamma(t)$ be a family of curves which evolves with normal velocity $v^\perp = V(t_{\gamma(t)}, k_{\gamma(t)})$, and for which $\gamma(0) = \gamma_0$ has no nodes. By theorem 1.4 $\gamma(t)$ won't have any nodes either. For such families we have the following result.

Theorem 8.1. *If V is $C^{2,1}$ and satisfies V_2, V_3, V_5^* and S, then any maximal classical solution of (0.1), which is simple and has no nodes, either exists for ever (i.e. $t_{Max} = \infty$), or else it shrinks to a point in finite time.*

Proof. Recall from section two that $V(t,k) = 0$ has a unique solution $k = K(t)$ for any $t \in S^1(M)$, and that this solution is uniformly bounded. If

$$A \stackrel{def}{=} \sup_{t \in S^1(M)} | K(t) | ,$$

then it follows from $V(t_{\gamma(t)}, k_{\gamma(t)}) \neq 0$ that either $k_{\gamma(t)} \geq -A$ holds for all t on the entire curve $\gamma(t)$, or else $k_{\gamma(t)} \leq A$ holds. We may assume that the first inequality holds.

The theorem now follows easily from the next lemma.

Lemma 8.2. *For any $A>0$ and any compact set $C \subset M$ there's an $\epsilon_{A,C}>0$ for which the following is true. If $\gamma \in \Omega(M)$ is a simple curve whose curvature is bounded from below by $k_\gamma \geq -A$, and if for some $Q \in C$ and $0<\epsilon<\epsilon_{A,C}$ there is a component of $\gamma \cap \mathbf{N}_\epsilon(Q)$ whose total curvature is at least $3\pi/2$, then $\gamma \subset \mathbf{N}_{2\epsilon}(Q)$.*

Proof of the lemma. If $A=0$ and M were flat, then γ would be a convex curve, and the "worst case" is drawn in figure 8.1. Clearly the hypotheses imply that the convex curve γ must be contained in $\mathbf{N}_{\sqrt{2}\epsilon}(Q)$.

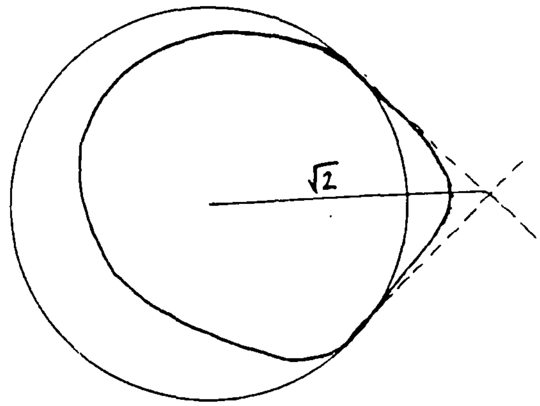

Figure 8.1 — A convex curve, with most of its curvature contained ina given circle.

In general we replace the metric g by $\hat{g}=\epsilon^{-2}g$. Since C is compact, the curvature and all its derivatives are bounded on C, so that we may assume that the disk $\hat{\mathbf{N}}_{10}(Q)=\mathbf{N}_{10\epsilon}(Q)$ is as close to Euclidean as we like, by making ϵ small. The condition $k_\gamma \geq -A$ gets replaced by $\hat{k}_\gamma \geq -\epsilon A$, so that for small enough ϵ any component of $\gamma \cap \hat{\mathbf{N}}_{10}(Q)$ will be almost convex.

So for small enough ϵ the situation is a small perturbation of the Euclidean case with $A=0$, and one sees that for any constant $c>\sqrt{2}$ (such as $c=2$) there's an $\epsilon_{A,C,c}>0$ such that $\gamma \subset \mathbf{N}_{c\epsilon}(Q)$ if $\epsilon<\epsilon_{A,C,c}$. $\square$

To complete the proof of the theorem, we choose a singular point Q on the limit curve γ^*, and we note that for any small $\epsilon>0$ the proof of theorem 7.1 shows that the total curvature of at least one component of $\gamma(t) \cap \mathbf{N}_\epsilon(Q)$ will exceed $3\pi/2$, for $t \geq t_\epsilon$. By the lemma we then get $\gamma(t) \subset \mathbf{N}_{2\epsilon}(Q)$. Since this works for any small $\epsilon>0$, we see that $\gamma(t)$ shrinks to the point Q. $\square$

References.

[A1] S. B. Angenent, *Abstract parabolic initial value problems,* preprint no.22, Leiden (1986).

[A2] S. B. Angenent, *The zeroset of a solution of a parabolic equation,* J. für die reine und angewandte Mathematik, **390** (1988), 79–96.

[AL] U. Abresch and J. Langer, *The normalized curve shortening flow and homothetic solutions,* J. Diff. Geom., **23** (1986), 175-196.

[DPG] G. Da Prato and P. Grisvard, *Equations d'evolution abstraites nonlineaires de type parabolique,* Ann. di Mat. Pura ed Appl. CXX (1979), 329-396.

[Ei] S. Eidelman, *Parabolic systems,* North Holland (1969).

[EW] C. Epstein and M. Weinstein, *A stable manifold theorem for the curve shortening equation,* Comm. Pure&Appl. Math., **40** (1987), 119-139.

[G]M. Gage, *Curve shortening on surfaces,* preprint.

[GH] M. Gage and R. Hamilton, *The shrinking of convex plane curves by the heat equation,* J. Diff. Geom., **23** (1986), 69-96.

[Gr1] M. Grayson, *The heat equation shrinks embedded plane curves to points,* J. Diff. Geom., 26 (1987), 285-314.

[Gr2] M. Grayson, *Shortening embedded curves,* preprint.

[LUS] O. A. Ladyzhenskaya, V. A. Solonikov and N. N. Ural'ceva, *Linear and quasilinear equations of parabolic type,* Translations of mathematical monographs **23**, AMS (1968).

[Ma] H. Matano, *Non increase of the lapnumber of a solution for a one dimensional semilinear parabolic equation*, J. Fac. Sci. Univ. Tokyo IA Math. **29**, No. 2 (1982), 401—441.

[RSK] J. Rubinstein, P. Sternberg, J. B. Keller, *Fast reaction, Slow diffusion, and curve shortening,* preprint, Stanford University (1987).

[St]C. Sturm, *Sur une classe d'equations a differences partielles,* J. Math. Pures Appl. **1** (1836), 373–444.